Course	:	**Game Theory**
Code	:	152180
Date	:	November 5, 2009
Time	:	08.45-11.45 hrs

This exam consists of 6 exercises. Motivate all your answers.

1. Two players face one stock supplied with seven pearls. Players move sequentially by removing either two or three pearls per move (unless a player is forced to remove the very last pearl). On the one hand, the reward to a player is determined by the total number of pearls removed per player, where the value of a single pearl for players 1 and 2 respectively are given by $\alpha \geq 0$ and $\beta \geq 0$. On the other hand, the player who removes the last pearl(s) receives an additional reward the amount of $\Delta \geq 0$ to be paid by the other player.

   (a) [2 pt] Depict the extensive form (tree structure) of this two-person non-cooperative game, in particular, equip each of five leaves with the rewards of both players.

   (b) [2 pt] Model this two-person non-cooperative game as a 3×4–bimatrix game (A, B) by specifying the pure strategy sets of both players and the corresponding payoff matrices A and B. Describe in words the three strategies of the (row) player 1 (who is supposed to do the very first move). The (column) player 2 has four strategies of the general form (i_2, i_3), that is, take i_k pearls at the first move if the (row) player 1 takes k pearls at his first move.

   (c) [2 pt] Suppose $\alpha = \beta = 0$ and $\Delta = 1$. Solve the corresponding 3×4–matrix game by determining its (upper-, lower-) value and the optimal pure strategies of both players.

2. For any $\alpha \in \mathbb{R}$, consider the following 2×2–bimatrix

$$(A, B) = \begin{bmatrix} (4,3) & (5,2) \\ (\alpha,2) & (4,3) \end{bmatrix}$$

   (a) [1 pt] Determine all the pure Nash Equilibria for all $\alpha \in \mathbb{R}$.

   (b) [2 pt] For all $\alpha \in \mathbb{R}$, determine the best reply set $BR_2(\vec{p})$ of the (column) player 2 against a mixed strategy $\vec{p} = (p_1, p_2) \in S^2$ of the (row) player 1 in the mixed extension of the bimatrix game (A, B). Depict $BR_2(\vec{p})$ as a multi-function of p_1.

   (c) [2 pt] For all $\alpha > 4$, determine the best reply set $BR_1(\vec{q})$ of the (row) player 1 against a mixed strategy $\vec{q} = (q_1, q_2) \in S^2$ of the (column) player 2 in the mixed extension of the bimatrix game (A, B). Depict $BR_1(\vec{q})$ as a multi-function of q_1.

   (d) [1 pt] For all $\alpha > 4$, determine all mixed Nash Equilibria.

3. Let $N = \{1,2,3\}$. The game (N, v) is given by:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	3	0	1	5	8	4	10

(a) [1.5 pt] Compute the core $C(v)$ and the domination core $DC(v)$. Write both sets as convex hull of their extreme points.

(b) [1.5 pt] Compute the Shapley value $\Phi(v)$ using the characterization with dividends.

(c) [3 pt] Compute the pre-nucleolus $\nu^*(v)$ and the nucleolus $\nu(v)$. Use the Kohlberg criterion to show that your answer is correct.

4. Let (N, v) be a game.

(a) [3 pt] Let $\mathbf{x} \in I(v)$ and $S \subseteq N$. Prove that the following statements are equivalent:
(i) There exists $\mathbf{z} \in I(v)$ with $\mathbf{z}\,\mathrm{dom}_S\,\mathbf{x}$
(ii) $x(S) < v(S)$ and $x(S) < v(N) - \sum_{i \in N \setminus S} v(\{i\})$.

(b) [3 pt] The dual game (N, v^*) of (N, v) is given by $v^*(S) = v(N) - v(N \setminus S)$, for every $S \subseteq N$. Prove that $\Phi(v^*) = \Phi(v)$.
Hint: First show that, if $v = \sum \alpha_T u_T$, then $v^* = \sum \alpha_T u_T^*$.

5. Consider the following discounted stochastic game with discount factor $\beta = 4/5$.

1 $(1,0,0)$	0 $(1,0,0)$
0 $(0,1,0)$	1 $(0,0,1)$

state 1

0 $(0,1,0)$

state 2

1 $(0,0,1)$

state 3

(a) [1 pt] Discounted stochastic games are a subclass of transient stochastic games. Under which conditions is a transient stochastic game a discounted stochastic game?

(b) [2 pt] Mention the three classes of strategies that a player may use. What kind of optimal strategies does a player have in a this game?

(c) [3 pt] Describe the approximation algorithm for discounted stochastic games. Perform one iteration of this algorithm.

(d) [3 pt] Calculate the value $\mathbf{v}_\beta$ and optimal strategies for the players. How good is your approximation from part (c)?

6. [3 pt] Consider a zero-sum discounted stochastic game with value vector $\mathbf{v}^*$. Let $\mathbf{f}^*$ be such that for all states s $\mathbf{f}^*(s)$ is an optimal action for player 1 in the matrix game

$$\left[(1-\beta)r(s,a^1,a^2)+\beta\sum_{s'\in\mathbf{S}}p(s'|s,a^1,a^2)v^*(s')\right]_{a^1=1,a^2=1}^{m^1(s),m^2(s)}$$

with value $v^*(s)$. Show that $v_\beta(\mathbf{f}^*,\mathbf{g})\geq\mathbf{v}^*$ for all $\mathbf{g}$.
Hint: use the following fact. If $\mathbf{v}$, $\mathbf{f}$, and $\mathbf{g}$ are such that $(1-\beta)\mathbf{r}(\mathbf{f},\mathbf{g})+\beta P(\mathbf{f},\mathbf{g})\mathbf{v}\geq\mathbf{v}$ then $v_\beta(\mathbf{f},\mathbf{g})\geq\mathbf{v}$.

Total: 36 + 4 points