Course	:	**Game Theory**
Code	:	191521800
Date	:	November 4, 2010
Time	:	08.45-11.45 hrs

This exam consists of 6 exercises. Motivate all your answers.

1. Consider the bimatrix game

$$(A, B) = \begin{pmatrix} 2,2 & 2,2 & 0,0 \\ 1,1 & 2,3 & 1,4 \\ 0,4 & 3,0 & 4,3 \end{pmatrix}.$$

(a) [1.5 pt] Determine all pure Nash equilibria of this bimatrix game.

(b) [1.5 pt] Explain how strict domination reduces the game (A, B) to the 2×2 game $\begin{pmatrix} 2,2 & 2,2 \\ 0,4 & 3,0 \end{pmatrix}$.

(c) [3 pt] Use the result from (b) to determine all Nash equilibria of the bimatrix game (A, B).

2. (a) [3 pt] An $m \times n$ matrix game $A = (a_{ij})$ is called symmetric if $m = n$ and $a_{ij} = -a_{ji}$ for all $i, j = 1, \ldots, m$.
Prove that the value of a symmetric game is zero, and that the sets of optimal strategies of players 1 and 2 coincide.

(b) [3 pt] Let (N, S_1, S_2, u_1, u_2) be a finite two-person game. Let $v_1 = x_1u_1 + y_1w$ and $v_2 = x_2u_2 + y_2w$, where $x_1, x_2 > 0$, $y_1, y_2 \in \mathbb{R}$ and $w : S_1 \times S_2 \to \mathbb{R}$ is the constant function on $S_1 \times S_2$ with value 1. Prove that the games (N, S_1, S_2, u_1, u_2) and (N, S_1, S_2, v_1, v_2) have the same set of Nash equilibria.

3. Let $N = \{1, 2, 3\}$. The game (N, v) is given by:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	4	1	5	3	6	12

(a) [1.5 pt] Compute the core $C(v)$. Write $C(v)$ as convex hull of its extreme points.

(b) [1.5 pt] Compute the Shapley value $\Phi(v)$ using the characterization based on potentials.

(c) [3 pt] Compute the pre-nucleolus $\nu^*(v)$. Use the Kohlberg criterion to show that your answer is correct.

4. (a) [3 pt] Consider a game (N,v) with $N = \{1,2,3\}$. Let $S_1, S_2, S_3 \subseteq N$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in I(v)$ be such that $\mathbf{x} \operatorname{dom}_{S_1} \mathbf{y}$, $\mathbf{y} \operatorname{dom}_{S_2} \mathbf{z}$ and $\mathbf{z} \operatorname{dom}_{S_3} \mathbf{x}$.

   (a1) [1.5 pt] Show that S_1, S_2 and S_3 are mutually distinct (i.e. $S_1 \neq S_2$, $S_1 \neq S_3$ and $S_2 \neq S_3$).

   (a2) [1.5 pt] Show that $C(v) = \varnothing$.

   (b) [3 pt] Prove that a game (N,v) is convex if and only if for all $T \in 2^N \backslash \{\varnothing\}$:

$$v(T) = \min_{\pi \in \Pi(N)} \sum_{i \in T} m_i^\pi(v).$$

5. Consider the following (zero-sum) discounted stochastic game with discount factor $\beta = 0.8$.

2 $(1,0)$	8 $(0,1)$
4 $(0,1)$	1 $(1,0)$

state 1

-1 $(0,1)$

state 2

   (a) [1 pt] Write down the set of equations that uniquely determine the value vector of the game.

   (b) [4 pt] Determine the value of this game and optimal strategies of the players.

6. (a) [2 pt] Mention two differences and two similarities between matrix games and zero-sum stochastic games.

   (b) [2 pt] Consider stochastic games with the average reward criterion. Assume that $\lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} P_{s_0\mathbf{fg}}[S_t = s]$ exists and equals $q(s)$, $s \in S$. Prove that $v_\alpha(s_0, \mathbf{f}, \mathbf{g}) = \sum_{s \in S} q(s) r(s, \mathbf{f}, \mathbf{g})$.

   (c) [3 pt] Let $(\mathbf{f}, \mathbf{g})$ be such that $P(\mathbf{f}, \mathbf{g})$ induces an irreducible Markov chain. Prove that if $v \in \mathbb{R}$ and $\mathbf{w} \in \mathbb{R}^N$ satisfy $\mathbf{w} + v\mathbf{1}_N = \mathbf{r}(\mathbf{f}, \mathbf{g}) + P(\mathbf{f}, \mathbf{g})\mathbf{w}$ then $v_\alpha(s, \mathbf{f}, \mathbf{g}) = v$ for any s.

Total: 36 + 4 points