Kenmerk: SK-158
Datum: 29 oktober 2004

Exam Measure and Probability (157040)
Thursday, 4 November 2004, 13.30 - 16.30 p.m.

This exam consists of 10 problems

1. Let Ω be a set, $\mathcal{F}$ a σ-field of subsets of Ω, and $\mu : \mathcal{F} \to [0, \infty]$ a function. When do we call
   a. μ a measure?
   b. $(\Omega, \mathcal{F}, \mu)$ a probability space?

2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that $\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$ for any sequence of sets $A_n \in \mathcal{F}$. Show by examples that the inequality may be strict or that it may be the equality.

3. Define Borel sets, and Borel measurable functions. Show that a monotone function on the real line is Borel.

4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $A \in \mathcal{F}$. Show that $\mathbf{1}_A = 0$ a.e. (almost everywhere) if and only if $\mu(A) = 0$. If $f : \Omega \to \mathbb{R}$ is a measurable function such that $f \neq 0$ a.e., show also that $f\mathbf{1}_A = 0$ a.e. implies $\mu(A) = 0$.

5. Suppose that f is a Lebesgue-measurable function such that $f \geq 0$.
   a. How is $\int_{\mathbb{R}} f dm$ defined?
   b. Show that $\int_{\mathbb{R}} f dm > 0$ if $m(\{x : f(x) > 0\}) > 0$.

6. Consider the probability space $([0, 1], \mathcal{M}_{[0,1]}, m_{[0,1]})$. Find F_X, the distribution function, and $\mathbb{E}(X)$, the expectation of
   a. a constant random variable X, $X(\omega) = a$ for all $\omega \in [0, 1]$;
   b. $X : [0, 1] \to \mathbb{R}$ given by $X(\omega) = \min\{\omega, 1 - \omega\}$ (the distance to the nearest endpoint of the interval $[0, 1]$);
   c. $X : [0, 1]^2 \to \mathbb{R}$, the distance to the nearest edge of the square $[0, 1]^2$.

P.T.O.

7. Let $f_{X,Y} = \mathbf{1}_A$, where A is the triangle with corners at $(0,0)$, $(2,0)$ and $(0,1)$. Find the conditional density $h(y|x)$ and conditional expectation $\mathbb{E}(Y|\, X = 1)$.

8. Let μ and ν be two finite measures on a measurable space $(\Omega, \mathcal{F})$ such that, for some $a > 0,\ b > 0$, we have $a\mu(A) \leq \nu(A) \leq b\mu(A)$ for all $A \in \mathcal{F}$. Show that μ and ν are equivalent measures (that is, $\mu \ll \nu$ and $\nu \ll \mu$) and that the respective Radon-Nikodym derivatives $f = d\nu/d\mu$ and $g = d\mu/d\nu$ satisfy $a \leq f \leq b$ μ-a.e. and $b^{-1} \leq g \leq a^{-1}$ ν-a.e.

9. Let F be an increasing, right-continuous function on $\mathbb{R}$, and m_F be the Lebesgue-Stieltjes measure corresponding to F.
   a. Show that, for any fixed $c > 0$, $\int_{\mathbb{R}}(F(x+c) - F(x))dx = cm_F(\mathbb{R})$.
   b. If F is a continuous function such that $\lim_{x\to-\infty} F(x) = 0$, show that $\int_{\mathbb{R}} F(x)dF(x) = (m_F(\mathbb{R}))^2/2$.

10. Consider a sequence of functions $f_n(x) = n^2 e^{-n|x|}$, $x \in \mathbb{R}$, and let $f(x) = 0$, $x \in \mathbb{R}$. Does f_n converge to f
   a. uniformly on $\mathbb{R}$?
   b. pointwise?
   c. almost everywhere?
   d. in L^p-norm?

1	2	3	4	5	6	7	8	9	10
2	3	3	2	3	3	2	3	3	3

Mark: $\frac{\text{Total}}{24} \times 9 + 1$ (rounded)