exam Measure and Probability 03-02-2006

3 1 a if (i) $\mu(\emptyset) = 0$ p.45

(ii) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$

(iii) $\mu(\bigcup_i A_i) \leq \sum_i \mu(A_i)$

b if (i) $\mu(A) \geq 0$

(ii) $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$ if $\{A_i\}$ disjoint

c if (i) μ is measure

(ii) $\mu(\Omega) = 1$

3 2 a. if $f^{-1}(I) \in \mathcal{M}$ for any interval I

b

$$1_A^{-1}((a,\infty)) = \begin{cases} \emptyset & \text{if } a \geq 1 \\ A & \text{if } 0 \leq a < 1 \\ \mathbb{R} & \text{if } a < 0 \end{cases}$$

hence 1_A is measurable $\Leftrightarrow A \in \mathcal{M}$

c. suppose $A \notin \mathcal{M}$ and let $f = 1_A - 1_{A^c}$,
then $|f| = 1$ while $f^{-1}((0,\infty)) = A \notin \mathcal{M}$

3 3 a if $X^{-1}((a,\infty)) \in \mathcal{B}_{[0,1]}$ for all $a \in \mathbb{R}$

b $P_X(B) = m_{[0,1]}(X^{-1}(B))$, $B \in \mathcal{B}$

c if $A \subset [0,1]$ then $m_{[0,1]}(A) = m(A) = \frac{1}{3} m(X(A))$
hence

$$P_X(B) = m_{[0,1]}(X^{-1}(B)) = m_{[0,1]}(X^{-1}(B \cap [-2,1]))$$

$$= m(X^{-1}(B \cap [-2,1])) = \frac{1}{3} m(B \cap [-2,1])$$

$$EX = \int_{-\infty}^{\infty} \tfrac{1}{3} x 1_{[-2,1]}(x)\, dx = \tfrac{1}{6} x^2 \Big|_{-2}^{1} = -\tfrac{1}{2}$$

b $F_X(x) = P(\{\omega : \min(\omega, 1-\omega) \leq x\}) = P(\{\omega : \omega \leq x \text{ or } \omega \geq 1-x\})$

$$= 1_{[\frac{1}{2},\infty)}(x) + 2x 1_{[0,\frac{1}{2})}(x), \quad x \in \mathbb{R}$$

$$f_X(x) = 2 1_{[0,\frac{1}{2})}(x)$$

$$EX = \int_{-\infty}^{\infty} 2x 1_{[0,\frac{1}{2})}(x)\, dx = x^2 \Big|_0^{1/2} = \tfrac{1}{4}$$

3 7 a.

$$P(X > Y) = \iint_{B} 1_A(x,y)\, dx\, dy = \int_0^{\frac{2}{3}} \int_y^{1-\frac{1}{2}y} dx\, dy = \tfrac{1}{3}$$

b. $f_X(x) = \int_{-\infty}^{\infty} 1_A(x,y)\, dy = 2(1-x) 1_{[0,1]}(x)$

$$f(y \mid X = x) = \frac{1_A(x,y)}{2(1-x)} = \frac{1_{[0,2(1-x)]}(y)}{2(1-x)} \qquad x \in (0,1)$$

c. $E(Y \mid X = x) = \int_{-\infty}^{\infty} y f(y \mid X = x)\, dy = 1 - x$

So $E(Y \mid X) = 1 - X$.

3 8 $X_n(\omega) \to \begin{cases} 0 & 0 < \omega < 1 \\ \infty & \omega = 0 \end{cases}$ as $n \to \infty$

Hence convergence a.s. and i.p., but not pointwise

$$\int_0^1 X_n(\omega)\, d\omega = \sqrt{n} \int_0^1 (1-\omega)^n d\omega = \frac{\sqrt{n}}{n+1} \to 0 \quad \text{in } L^1\text{-norm}$$

$$\int_0^1 X_n^2(\omega)\, d\omega = \frac{n}{2n+1} \not\to 0 \quad \text{not in } L^2\text{-norm}$$

3 4 a

$$\int_{\mathbb{R}} f\,dm = \sup\left\{ \int_{\mathbb{R}} \varphi\,dm : 0 \le \varphi \le f,\ \varphi \text{ simple} \right\}$$

if φ is simple with values $\{a_1, \dots, a_n\}$ and $A_i = \varphi^{-1}(\{a_i\})$ then

$$\int_{\mathbb{R}} \varphi\,dm = \sum a_i m(A_i)$$

b let $E_n = f^{-1}([\frac{1}{n}, \infty))$, then $E = \bigcup_n E_n = \{x : f(x) > 0\}$

$m(E) > 0 \implies m(E_n) > 0$ for some $n = n^*$

let $\varphi = \frac{1}{n^*} 1_{E_{n^*}}$, then

$$\int_{\mathbb{R}} f\,dm \ge \int_{\mathbb{R}} \varphi\,dm = \frac{1}{n^*} m(E_{n^*}) > 0$$

3 5 a if $|f_n| \le g$ a.e., $f = \lim_{n\to\infty} f_n$ a.e. and $f, g \in \mathcal{L}^1$ then

$$\lim_{n\to\infty} \int_E f_n\,dm = \int_E f\,dm$$

b. $$\frac{1}{(1+\frac{x}{n})^n \sqrt[n]{x}} < \frac{1}{\sqrt[n]{x}} < \frac{1}{\sqrt{x}} \in \mathcal{L}^1_{[0,1]}, \quad x \in (0,1)$$

hence

$$\lim_{n\to\infty} \int_0^1 \frac{1}{(1+\frac{x}{n})^n \sqrt[n]{x}}\,dx \overset{\text{d.c.}}{=} \int_0^1 e^{-x}\,dx = 1 - e^{-1}$$

4 6 a $$F_X(x) = P(\{\omega : 3\omega - 2 \le x\}) = P(\{\omega : \omega \le \tfrac{1}{3}(x+2)\})$$

$$= 1_{[1,\infty)}(x) + \tfrac{1}{3}(x+2)\, 1_{[-2,1]}(x), \quad x \in \mathbb{R}$$

$$f_X(x) = \tfrac{1}{3} 1_{[-2,1]}(x), \quad x \in \mathbb{R}$$