1 a $\mathcal{F} \subset \mathcal{P}(\Omega)$ is σ-field if

(i) $\Omega \in \mathcal{F}$

2 (ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_n \in \mathcal{F} \Rightarrow \bigcup_n A_n \in \mathcal{F}$

b (i) $B^c \subset \Omega$ so $\Omega \in \mathcal{G}$

(ii) $A \in \mathcal{G} \Leftrightarrow A \subset B$ or $B^c \subset A$

2 $\Leftrightarrow B^c \subset A^c$ or $A^c \subset B$

$\Leftrightarrow A^c \in \mathcal{G}$

(iii) $A_n \in \mathcal{G}$

$\forall_n \ A_n \subset B \Rightarrow \bigcup_n A_n \subset B \Rightarrow \bigcup_n A_n \in \mathcal{G}$

$\exists_n \ B^c \subset A_n \Rightarrow B^c \subset \bigcup_n A_n \Rightarrow \bigcup_n A_n \in \mathcal{G}$

1 c $f^{-1}(I) \in \mathcal{F}$ for any interval I.

2 a if μ is a measure, i.e. if, for pairwise disjoint sets E_i

1 $$\mu\left(\bigcup_i E_i\right) = \sum_i \mu(E_i)$$

b Let $B_1 = A_1$, $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_n$, then

2 $$\bigcup_n A_n = \bigcup_n B_n \text{ and } \mu(B_n) \leq \mu(A_n)$$

hence

$$\mu\left(\bigcup_n A_n\right) = \mu\left(\bigcup_n B_n\right) = \sum_n \mu(B_n) \leq \sum_n \mu(A_n)$$

strict inequality:

1 $A_n = A$, $\mu(A) > 0$

strict equality:

pairwise disjoint A_n

3 probability space: $(\Omega, \mathcal{F}, P)$

1 $\Omega := \{ (n_A, n_B) : \; n_A, n_B \in \{1, 2, \dots, 6\}\}$

$\mathcal{F} := \mathcal{P}(\Omega) \qquad P: \quad P(A) = |A|/36$

r.v. X:

1 $$X((n_A, n_B)) = \begin{cases} n_B & \text{if } n_A \text{ even} \\ -n_A & \text{if } n_A \text{ odd} \end{cases}$$

$$EX = \int_\Omega X \, dP$$

$$= \sum_{k=1}^{6} k \frac{3}{36} + (-1)\frac{6}{36} + (-3)\frac{6}{36} + (-5)\frac{6}{36} = \frac{1}{4}$$

4 a MCT: If $f_n \geq 0$ measurable and $f_n \uparrow f$ pointwise then

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm$$

b. s_n simple hence measurable and $s_n \in \mathcal{L}^1(E)$

$\int f \, dm \leq \int s_n \, dm < \infty$ hence $f \in \mathcal{L}^1(E)$

$s_n \downarrow f \Rightarrow s_1 - s_n \geq 0$ and $s_1 - s_n \uparrow s_1 - f \geq 0$

?

MCT: $\lim\limits_{n\to\infty} \int_E (s_1 - s_n) \, dm = \int_E (s_1 - f) \, dm$

$$\Rightarrow \lim_{n\to\infty} \int_E s_n \, dm = \int_E f \, dm \qquad \text{since } \int_E s_1 \, dm < \infty$$

5 a. DCT: If f_n measurable, $|f_n| \le g$ a.e., $g \in \mathcal{L}^1$ and $f = \lim_{n\to\infty} f_n$ a.e, then $f \in \mathcal{L}^1$ and

2

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm$$

b. $f_n(x) = e^{-nx^2}$ then $|f_n| \le 1$ and

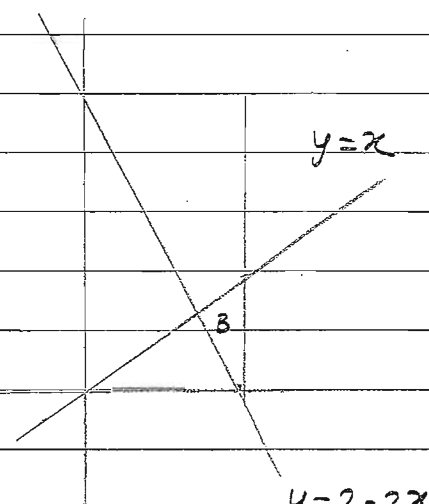
1

$f_n \to 0$ a.e., so (DCT)

$$\lim_{n\to\infty} \int_0^1 e^{-nx^2} dx = 0$$

6

a $P(X > Y) = \iint_B 1_A(x,y)\, dx\, dy$



$$= \int_{\frac{2}{3}}^{1} \int_{2-2x}^{x} dy\, dx$$

1

$$= \int_{\frac{2}{3}}^{1} (3x-2) = \frac{1}{6}$$

b $f_Y(y) = \int_{-\infty}^{\infty} 1_A(x,y)\, dx = \frac{1}{2} y \, 1_{[0,2]}(y)$

1

$f(x \mid Y=y) = \frac{2}{y} 1_A(x,y) = \frac{2}{y} 1_{[1-\frac{1}{2}y,1]}(x)$, $y \in (0,2)$

c $E(X \mid Y=y) = \int_{-\infty}^{\infty} x f(x \mid Y=y)\, dx = 1 - \frac{1}{4}y$, $y \in (0,2)$

1

$\therefore E(X \mid Y) = 1 - \frac{1}{4} Y$

7 a. $\nu \ll \mu \iff \forall_{A \in \mathcal{F}} \; \mu(A) = 0 \Rightarrow \nu(A) = 0$

b. RNT: If $\nu \ll \mu$ then there exists a measurable function h such that for all $A \in \mathcal{F}$,

$$\nu(A) = \int_A h \, d\mu$$

c. Choosing

$$h(\omega) = \begin{cases} \nu(\omega)/\mu(\omega) & \text{if } \mu(\omega) > 0 \\ 0 & \text{else} \end{cases}$$

we have

$$\int_A h \, d\mu = \sum_{\substack{\omega \in A \\ \mu(\omega) > 0}} \nu(\omega) = \nu(A)$$

hence $\frac{d\nu}{d\mu} = h$.

8 Since $f_n(0) = n^2 \to \infty$, there is no pointwise, and hence no uniform convergence.

$f_n(x) \to 0$ if $x \neq 0$ so conv. a.e.

$$\int_{\mathbb{R}} |f_n|^p \, dm = 2 \int_0^\infty n^{2p} e^{-npx} \, dx$$

$$= 2n^{2p} \cdot \frac{1}{np} = \frac{2}{p} n^{2p-1} \to 0 \quad \text{iff } p < \tfrac{1}{2}$$

so no convergence in $\mathcal{L}^p$-norm if $p \geq 1$