Kenmerk: EWI/TW/SP/028
Datum: 9 april 2009

Exam Measure and Probability (157040)
Thursday, 16 April 2009, 13.30 - 16.30 p.m.

This exam consists of 7 problems

1. a. Define what is meant by $m^*(A)$, the Lebesgue outer measure of $A \subset \mathbb{R}$.

   b. Use the countable subadditivity of Lebesgue outer measure to show that $m^*(A) = 0$ implies $m^*(A \cup B) = m^*(B)$ for each $B \subset \mathbb{R}$.

   c. Define what is meant by saying that $A \subset \mathbb{R}$ is measurable.

2. a. Define what is meant by saying that $f : \mathbb{R} \to \mathbb{R}$ is measurable.

   b. Show that every monotone function $f : \mathbb{R} \to \mathbb{R}$ is measurable.

   c. Show that $f : \mathbb{R} \to \mathbb{R}$ is measurable if and only if $\{x \in \mathbb{R} : f(x) > r\}$ is measurable for each rational number r.

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *monotone convergence theorem.*

   b. Let $f : \mathbb{R} \to \mathbb{R}$ and $f \geq 0$, and define $\nu : \mathcal{M} \to \mathbb{R}$ by

$$\nu(A) = \int_A f dm, \quad A \in \mathcal{M}.$$

   Show that ν is a measure. (Hint: use the monotone convergence theorem.)

4. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *dominated convergence theorem.*

   b. Evaluate

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin(x/n) dx.$$

5. Consider the probability space $([0, 1], \mathcal{M}_{[0,1]}, m_{[0,1]})$. Find F_X, the distribution function, and $\mathbb{E}(X)$, the expectation of

   a. the random variable X, given by $X(\omega) = 2\omega - 1$,

   b. the random variable X given by $X(\omega) = \max(\omega, 1 - \omega)$.

6. Let X and Y be two random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with joint density

$$f_{X,Y}(x,y) = \mathbf{1}_A(x,y), \quad (x,y) \in \mathbb{R}^2,$$

where A is the triangle with corners at (0,2), (1,0) and (1,2).

a. Find $P(X \leq Y)$.

a. Find the conditional density $f_{X|Y}(x|Y=y)$ of X given $Y = y$.

b. Determine $E(X|Y)$.

7. Consider the probability space $((0,1), \mathcal{M}_{(0,1)}, m_{(0,1)})$ and, for $n = 1, 2, \ldots$, set

$$X_n(\omega) = \begin{cases} n & \text{if } \; 0 < \omega < \frac{1}{n} \\ 0 & \text{if } \; \frac{1}{n} < \omega < 1. \end{cases}$$

Which of the following statements are true? (Justify your answers).

a. $X_n \to 0$ in probability.

b. $X_n \to 0$ almost surely.

c. $X_n \to 0$ pointwise.

d. $X_n \to 0$ in L^1-norm.

e. $X_n \to 0$ in L^2-norm.

1	2	3	4	5	6	7
4	4	4	4	4	4	3

Mark: $\frac{\text{Total}}{27} \times 9 + 1$ (rounded)