1. a. $\Omega \in \mathcal{F}$

2 $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

$A_n \in \mathcal{F} \Rightarrow \bigcup_n A_n \in \mathcal{F}$

b $\mathcal{F}$ σ-field

2 $\mu(A) \geq 0$

$\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$ if $\{A_n\}$ disjoint

c $\mathcal{F}$ σ-field

1 μ measure

$\mu(\Omega) = 1$

1 d $f^{-1}(I) \in \mathcal{F}$ for any interval I

e

2 let $I = (-\infty, x]$, then $f^{-1}(I) = \{\omega : f(\omega) \leq x\}$

$f(\omega) \leq x \iff \forall_{\varepsilon > 0} \exists_N \forall_{n \geq N} [f_n(\omega) \leq x + \varepsilon]$

$$\therefore \{\omega : f(\omega) \leq x\} = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega : f_n(\omega) \leq x + \tfrac{1}{m}\}$$

$$= \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} f_n^{-1}((-\infty, x + \tfrac{1}{m}]) \in \mathcal{F}$$

2 a if $f_n \geq 0$ and $f_n \uparrow f$ pointwise, then

$$\lim_{n \to \infty} \int_E f_n \, dm = \int_E f \, dm.$$

b let $g_n = \sum_{k=1}^{n} f_k$, then $g_n \uparrow g$ and $g_n \geq 0$.

By the MCT

$$\int g \, dm = \int \lim_{n\to\infty} g_n \, dm = \lim_{n\to\infty} \int g_n \, dm$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k \, dm = \sum_{k=1}^{\infty} \int f_k \, dm$$

3 a if $|f_n| \leq g$ a.e., $f = \lim_{n\to\infty} f_n$ a.e. and $g \in \mathcal{L}^1(E)$
then $f \in \mathcal{L}^1(E)$,

2 $$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

b. $$|f_n(x)| = \left| \frac{1+nx^2}{(1+x^2)^n} \right| \leq 1 \in \mathcal{L}^1([0,1])$$

2 $$\lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n} = 1\!\!1_{\{0\}}(x) = 0 \text{ a.e.}$$

by the DCT:
$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n} = 0$$

4 a. if f integrable w.r.t. $\mu_1 \times \mu_2$

1 $$\Longleftrightarrow \int_{\Omega_1} \int_{\Omega_2} |f| \, d\mu_2 \, d\mu_1 < \infty$$

b $$\int_{\mathbb{R}} \left(F(x+a) - F(x) \right) dx = \int_{\mathbb{R}} \int_{\mathbb{R}} 1\!\!1_{(x,x+a]}(y) \, dP(y) \, dm(x)$$

2 $$= \int_{\mathbb{R}} \int_{\mathbb{R}} 1\!\!1_{(x,x+a]}(y) \, dm(x) \, dP(y)$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} 1\!\!1_{[y-a,y)}(x) \, dm(x) \, dP(y) = \int_{\mathbb{R}} a \, dP(y) = a$$

5 a $\quad F_X(x) = P(\{\omega : a \leq x\}) = \mathbb{1}_{[a,\infty)}(x)$

1 $\quad EX = \int_0^1 a \, dm_{[0,1]}(\omega) = a$

b $\quad F_X(x) = P(\{\omega : \frac{1}{1+\omega} \leq x\})$

$$= \left(2 - \frac{1}{x}\right) \mathbb{1}_{[\frac{1}{2},1)}(x) + \mathbb{1}_{[1,\infty)}(x)$$

2 $\quad EX = \int_0^1 \frac{1}{1+\omega} \, dm_{[0,1]}(\omega) = \ln 2$

or via f_X

1 $\quad$ 6 a $\quad P_2 \ll P_1 \iff \forall_{A \in \mathcal{B}} \, [P_1(A) = 0 \Rightarrow P_2(A) = 0]$

b $\quad$ if $P_2 \ll P_1$ then there exists a measurable function h such that for all $A \in \mathcal{B}$

1 $$P_2(A) = \int_A h \, dP_1 \qquad \left(h = \frac{dP_2}{dP_1}\right)$$

c $\quad P_1 \ll P_3, \; P_2 \ll P_3$

$$P_1(A) = \int_A \frac{dP_1}{dP_3} \, dP_3 = \frac{1}{2} \int_A \frac{dP_1}{dP_3} \, dP_1 + \frac{1}{2} \int_A \frac{dP_1}{dP_3} \, dP_2$$

2 $$\iff \frac{1}{2}\left(\mathbb{1}_A(0) + \mathbb{1}_A(10)\right) = \frac{1}{4} \frac{dP_1}{dP_3}(0) \mathbb{1}_A(0) + \frac{1}{4} \frac{dP_1}{dP_3}(10) \mathbb{1}_A(10) + \frac{1}{20} \int_A \frac{dP_1}{dP_3}(x) \, dm_{[0,10]}$$

$$\therefore \quad \frac{dP_1}{dP_3}(x) = 2 \, \mathbb{1}_{\{0,10\}}(x)$$

similarly $\quad \frac{dP_2}{dP_3}(x) = 2 \, \mathbb{1}_{(0,10)}(x)$

7 a $F_n(x) = P(X_n \leq x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} - \frac{1}{2n} & 0 \leq x < \frac{1}{n} \\ 1 - \frac{1}{2n} & \frac{1}{n} \leq x < n \\ 1 & x \geq n \end{cases}$

/

b. $X_n \to 0$ pointwise, and hence a.s. and i.p.

$X_n \to 0$ weakly since $F_n(x) \to 1\!\!1_{[0,\infty)}(x)$ a.e.

3

$\forall_n \exists_\omega [X_n \geq 1]$, so no uniform conv.

$\int_0^1 |X_n| = \frac{1}{2n} + \frac{1}{2} \to \frac{1}{2} \neq 0$, so no conv. in $\mathcal{L}^1$ norm