Kenmerk: EWI10/TW/SP/002
Datum: 7 januari 2010

Exam Measure and Probability (157040)
Monday, 18 January 2010, 8.45 - 11.45 a.m.

This exam consists of 7 problems

1. Let Ω be a set, $\mathcal{F}$ a collection of subsets of Ω, $\mu : \mathcal{F} \to \mathbb{R}$, and $f : \Omega \to \mathbb{R}$. When do we call

   a. $\mathcal{F}$ a *σ-field*?

   b. μ a *measure*?

   c. $(\Omega, \mathcal{F}, \mu)$ a *probability space*?

   Suppose that $\mathcal{F}$ is a σ-field.

   d. What does it mean to say that f is *$\mathcal{F}$-measurable*?

   e. Let f_n be a sequence of $\mathcal{F}$-measurable functions such that $f_n(\omega) \to f(\omega)$ as $n \to \infty$ for each $\omega \in \Omega$. Show that the function f is $\mathcal{F}$-measurable.

2. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *monotone convergence theorem.*

   b. Let $\{f_n\}_{n\geq 1}$ be a sequence of non-negative measurable functions and define $g = \sum_{n=1}^{\infty} f_n$. Show that

$$\int g dm = \sum_{n=1}^{\infty} \int f_n dm.$$

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *dominated convergence theorem.*

   b. Evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n} dx.$$

4. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces, and let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$.

   a. Under which condition do we have

$$\int_{\Omega_1} \left(\int_{\Omega_2} f d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} f d\mu_1 \right) d\mu_2?$$

   (This is *Fubini's Theorem*.)

b. Use Fubini's Theorem to show that for any distribution function F

$$\int_{\mathbb{R}} (F(x+a) - F(x))dx = a.$$

5. Consider the probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$. Find F_X, the distribution function, and $\mathbb{E}(X)$, the expectation of

   a. a constant random variable X, $X(\omega) = a$ for all $\omega \in [0,1]$;

   b. the random variable X given by $X(\omega) = \frac{1}{1+\omega}$.

6. Let P_1, P_2 and P_3 be probability measures on $(\mathbb{R}, \mathcal{B})$.

   a. What is meant by $P_2 \ll P_1$ (P_2 is *absolutely continuous with respect to* P_1)?

   b. What does the *Radon-Nikodym Theorem* say about the relation between P_1 and P_2 if $P_2 \ll P_1$?

   c. Let $P_1 = \frac{1}{2}(\delta_0 + \delta_{10})$, $P_2 = \frac{1}{10} m_{[0,10]}$ and $P_3 = \frac{1}{2}P_1 + \frac{1}{2}P_2$. For which $i \neq j$ do we have $P_i \ll P_j$? Find the Radon-Nikodym derivative in each such case.

7. Consider the probability space $([0,1), \mathcal{M}_{[0,1)}, m_{[0,1)})$ and, for $n = 1, 2, \ldots$, set

$$X_n(\omega) = \begin{cases} 0 & \text{if} \quad 0 \leq \omega < \frac{1}{2} - \frac{1}{2n} \\ n & \text{if} \quad \frac{1}{2} - \frac{1}{2n} \leq \omega < \frac{1}{2} \\ \frac{1}{n} & \text{if} \quad \frac{1}{2} \leq \omega < 1. \end{cases}$$

   a. Find the distribution function $F_n(x)$ of X_n.

   b. Which of the following statements are true? (Justify your answers).

      (i) $X_n \to 0$ in probability.

      (ii) $X_n \to 0$ weakly.

      (iii) $X_n \to 0$ almost surely.

      (iv) $X_n \to 0$ pointwise.

      (v) $X_n \to 0$ in L^1-norm.

      (vi) $X_n \to 0$ uniformly.

1	2	3	4	5	6	7
8	4	4	3	3	4	4

Mark: $\frac{\text{Total}}{30} \times 9 + 1$ (rounded)