Kenmerk: EWI/TW/SP/11-031
Datum: 28 maart 2011

Exam Measure and Probability (191570401)
Monday 4 April 2011, 13.45 - 16.45 p.m.

This exam consists of 7 problems

1. Let Ω be a set, $\mathcal{F}$ a σ-field of subsets of Ω, and $\mu : \mathcal{F} \to \mathbb{R}$. When do we call
   a. μ an outer measure?
   b. μ a measure?
   c. $(\Omega, \mathcal{F}, \mu)$ a probability space?

2. Suppose $E \subset \mathbb{R}$ is a (Lebesgue-)measurable set.
   a. Define what is meant by saying that $f : E \to \mathbb{R}$ is measurable.
   b. Show that $f : E \to \mathbb{R}$ is measurable if and only if $\{x \in E : f(x) > r\}$ is measurable for each rational number r. (Hint: for each $a \in \mathbb{R}$ there is a decreasing sequence of rational numbers r_n such that $a = \lim_{n\to\infty} r_n$.)
   c. Use the result of b. to show that $\{x \in E : f_1(x) > f_2(x)\}$ is measurable if $f_1 : E \to \mathbb{R}$ and $f_2 : E \to \mathbb{R}$ are measurable.

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$. Let $f : \mathbb{R} \to \mathbb{R}$ and $f \geq 0$, and define $\nu : \mathcal{M} \to \mathbb{R}$ by
$$\nu(A) = \int_A f dm, \quad A \in \mathcal{M}.$$
   a. State the *monotone convergence theorem.*
   b. Show that ν is a measure. (Hint: use the monotone convergence theorem.)

4. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.
   a. State the *dominated convergence theorem.*
   b. Evaluate
$$\lim_{n\to\infty} \int_0^1 e^{-nx^2} dx.$$

5. Consider the probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$. Find F_X, the distribution function, and $\mathbb{E}(X)$, the expectation of

   a. the random variable X, given by $X(\omega) = 2\omega - 1$,

   b. the random variable X given by $X(\omega) = \max(\omega, 1-\omega)$.

6. Let $(\Omega, \mathcal{F})$ be a measurable space and let μ and ν be finite measures on $(\Omega, \mathcal{F})$.

   a. What is meant by $\nu \ll \mu$ (ν is *absolutely continuous with respect to* μ)?

   b. What does the *Radon-Nikodym Theorem* say about the relation between μ and ν if $\nu \ll \mu$?

   c. Give the *Radon-Nikodym derivative* $\frac{d\nu}{d\mu}$ if $\nu \ll \mu$ and Ω is finite.

7. Consider the probability space $((0,1), \mathcal{M}_{(0,1)}, m_{(0,1)})$ and, for $n = 1, 2, \ldots$, set

$$X_n(\omega) = \begin{cases} n & \text{if } 0 < \omega < \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < \omega < 1. \end{cases}$$

   Which of the following statements are true? (Justify your answers).

   a. $X_n \to 0$ in probability.

   b. $X_n \to 0$ almost surely.

   c. $X_n \to 0$ pointwise.

   d. $X_n \to 0$ in L^1-norm.

   e. $X_n \to 0$ in L^2-norm.

1	2	3	4	5	6	7
3	3	2	2	2	3	3

Mark: $\frac{\text{Total}}{18} \times 9 + 1$ (rounded)