UNIVERSITEIT TWENTE

Faculteit Elektrotechniek, Wiskunde en Informatica

Exam Measurability and Probability (1915703401) on Monday, January 18, 2016, 8.45 – 11.45 hours.

The solutions of the exercises should be clearly formulated and clearly written down. Moreover, you should in all cases include a convincing argument with your answer.

With this exam a calculator is **not** permitted. Also a formula sheet is **not** permitted.

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1. Let $\Omega = [0,3]$. Let $\mathcal{F}$ be the smallest σ-algebra such that $X : \Omega \to \mathbb{R}$ defined by

   $$X(t) = \begin{cases} t & t \in [0,1] \\ 1 & t \in (1,2) \\ t-1 & t \in [2,3] \end{cases}$$

   is measurable. Show that $Y : \Omega \to \mathbb{R}$ defined by:

   $$Y(t) = t$$

   is not measurable with respect to $\mathcal{F}$.

   Hint: For any set A, if $1 \in X^{-1}(A)$ then $2 \in X^{-1}(A)$.

2. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a) State the *monotone convergence theorem.*

   b) Let $\{f_n\}_{n\geq 1}$ be a sequence of non-negative measurable functions and define $g = \sum_{n=1}^{\infty} f_n$. Show that

   $$\int g dm = \sum_{n=1}^{\infty} \int f_n dm.$$

3. Investigate the convergence of:

   $$\lim_{n\to\infty} \int_0^a \frac{n + \sin x}{(x+n)(x+2n)} \mathrm{d}x$$

   for any finite a. If you use any theorems from the book then clearly formulate that theorem.

4. Let $([0,1], \mathcal{M}_{[0,1]}, P)$ be a measurable space such that for $g : [0,1] \to \mathbb{R}$ with $g(\omega) = \omega$ we have:

   $$P(A) = \int_A g dm$$

   Let $\Omega_2 = [0,1] \times [0,1]$, $P_2 = P \times P$ and $\mathcal{M}_2 = \mathcal{M}_{[0,1]} \times \mathcal{M}_{[0,1]}$. Let $f : \Omega_2 \to \mathbb{R}$ be defined by:

   $$f(x,y) = x^2 - y^2$$

   Compute:

   $$\int_{\Omega_2} f \, \mathrm{d}P_2$$

   If you use any theorems from the book then clearly formulate that theorem.

see reverse side

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5. Consider the interval $[-1,1]$ with Lebesgue measure $m_{[-1,1]}$ and let ν and μ be measures on the measurable space $([-1,1], \mathcal{B}_{[-1,1]})$ such that

$$\nu([-1,x]) = \begin{cases} 0 & -1 \le x < 0, \\ 1+x^2 & 0 \le x \le 1, \end{cases} \qquad \mu([-1,x]) = \begin{cases} 0 & -1 \le x < 0, \\ x^2 & 0 \le x \le 1. \end{cases}$$

   a) Verify whether ν and μ are absolutely continuous with respect to $m_{[-1,1]}$.
   b) Determine the Radon-Nikodym derivatives of ν and μ with respect to $m_{[-1,1]}$ if they exist.

6. Let X and Y be two random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ such that

$$P(X \le x, Y \le y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ ax^2 + by^2 + cxy & x \in [0,1] \text{ and } y \in [0,1] \\ a + by^2 + cy & x > 1 \text{ and } y \in [0,1] \\ ax^2 + b + cx & x \in [0,1] \text{ and } y > 1 \\ 1 & x > 1 \text{ and } y > 1 \end{cases}$$

   for certain values of a, b, c.

   a) For which values of a, b and c are X and Y both well-defined stochastic variables.
   b) For which values of a, b and c are X and Y both well-defined stochastic variables and additionally X has a well-defined density.
   c) For which values of a, b and c are X and Y both well-defined stochastic variables with a well-defined joint density.

7. Consider the probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$ and set

$$X_n(\omega) = \max\left\{n - n^2|\omega - \frac{1}{n}|, 0\right\}, \quad n = 1, 2, \ldots.$$

   a) Does X_n converge to 0 uniformly? Pointwise?
   b) Does X_n converge to 0 almost surely? In probability?
   c) Does X_n converge to 0 in L^1-norm?

   Motivate your answers.

For the questions the following number of points can be awarded:

Exercise	1.	8 points	Exercise	4.	8 points	Exercise	7.	8 points
Exercise	2.	7 points	Exercise	5.	8 points			
Exercise	3.	7 points	Exercise	6.	8 points			

The final grade is determined by adding 6 points to the total number of points awarded and dividing by 6.