UNIVERSITEIT TWENTE

Faculteit Elektrotechniek, Wiskunde en Informatica

Exam Measurability and Probability (1915703401) on Wednesday, April 6, 2016, 13:45 - 16:45 hours.

The solutions of the exercises should be clearly formulated and clearly written down. Moreover, you should in all cases include a convincing argument with your answer.
With this exam a calculator is **not** permitted. Also a formula sheet is **not** permitted.

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1. Let $\Omega = [0, \infty)$. Let $\mathcal{F}$ be the smallest σ-algebra such that

$$\left[\frac{p}{2}, \frac{p}{2} + 1\right) \in \mathcal{F}$$

for $p = 0, 1, 2, \ldots$.

a) Show that $\left[0, \frac{1}{2}\right) \in \mathcal{F}$.

Let μ be a measure defined on $\mathcal{F}$ such that

$$\mu\left(\left[\frac{p}{2}, \frac{p}{2} + 1\right]\right) = \frac{3}{2^{p+1}}$$

for $p = 0, 1, 2, \ldots$.

b) Determine

$$\mu\left(\left[0, \frac{1}{2}\right)\right).$$

2. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$. Investigate the convergence of:

$$\lim_{n \to \infty} \int_0^\infty \frac{x + \sin x + 2n}{x^2 + nx} \, dm$$

If you use any theorems from the book then clearly formulate that theorem.

3. Investigate the convergence of:

$$\lim_{n \to \infty} \int_0^3 \frac{x + n + \sin(nx)}{x + n} \, dx$$

If you use any theorems from the book then clearly formulate that theorem.

4. Let X and Y be two random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with joint density

$$f_{X,Y}(x, y) = \mathbf{1}_A(x, y), \quad (x, y) \in \mathbb{R}^2,$$

where A is the triangle with corners at $(0, 2)$, $(1, 0)$ and $(1, 2)$.

a) Find $P(X > Y)$.

b) Find the conditional density $f_{X|Y}(x|Y = y)$ of X given $Y = y$.

c) Determine $E(X|Y)$.

see reverse side

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5. Let μ and ν be two finite measures on a measurable space $(\Omega, \mathcal{F})$.

   a) What is meant by $\mu(A) \ll \nu(A)$ (μ is *absolutely continuous with respect to* ν)?

   Suppose that, for some $a > 0$, $b > 0$, we have $a\mu(A) \leq \nu(A) \leq b\mu(A)$ for all $A \in \mathcal{F}$.

   b) Show that μ and ν are equivalent measures (that is, $\mu \ll \nu$ and $\nu \ll \mu$).

   c) Show that the respective Radon-Nikodym derivatives $f = \mathrm{d}\nu/\mathrm{d}\mu$ and $g = \mathrm{d}\mu/\mathrm{d}\nu$ satisfy $a \leq f \leq b$ μ-a.e. and $b^{-1} \leq g \leq a^{-1}$ ν-a.e.

6. Consider the probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$. Find F_X, the distribution function, and $\mathbb{E}(X)$, the expectation of

   a) the random variable X, given by $X(\omega) = \max(\omega, 1 - 2\omega)$,

   b) the random variable X given by $X(\omega) = \min(\omega, 1 - \omega^2)$.

7. Consider the probability space $([0,1), \mathcal{M}_{[0,1)}, m_{[0,1)})$ and, for $n \in \mathbb{N}$, set

$$X_n(\omega) = \begin{cases} 0 & \text{if } 0 \leq \omega < \frac{1}{2} - \frac{1}{2n} \\ n & \text{if } \frac{1}{2} - \frac{1}{2n} \leq \omega < \frac{1}{2} \\ \frac{1}{n} & \text{if } \frac{1}{2} \leq \omega < 1. \end{cases}$$

   Which of the following statements are true? (Justify your answers).

   a) $X_n \to 0$ in probability.

   b) $X_n \to 0$ weakly.

   c) $X_n \to 0$ almost surely.

   d) $X_n \to 0$ pointwise.

   e) $X_n \to 0$ in L^1-norm.

   f) $X_n \to 0$ uniformly.

For the questions the following number of points can be awarded:

Exercise	1.	8 points	Exercise	4.	8 points	Exercise	7.	8 points
Exercise	2.	7 points	Exercise	5.	8 points			
Exercise	3.	7 points	Exercise	6.	8 points			

The final grade is determined by adding 6 points to the total number of points awarded and dividing by 6.