Exam Measure and Probability (191570401)
Monday 22 January 2018, 8.45 - 11.45 a.m.

This exam consists of 6 problems

1. Let Ω be a set, $\mathcal{F}$ a collection of subsets of Ω, and $\mu : \mathcal{F} \to [0, \infty)$. When do we call

   a. $\mathcal{F}$ a σ-field?

   b. μ an outer measure?

   c. μ a measure?

   Suppose $\mathcal{F}$ is a σ-field and μ a *finitely-additive* set function, that is, $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever A and B are disjoint sets in $\mathcal{F}$. Also suppose μ has the following property: If $E_1 \supset E_2 \supset E_3 \supset \ldots$ are sets in $\mathcal{F}$ such that $\cap_i E_i = \emptyset$, then $\lim_{i\to\infty} \mu(E_i) = 0$.

   d. Prove that μ is a measure on $(\Omega, \mathcal{F})$.

2. Consider the measure space $((0,1), \mathcal{M}_{(0,1)}, m_{(0,1)})$.

   a. What is meant by saying that $f : (0,1) \to \mathbb{R}$ is measurable?

   b. What is meant by saying that $f : (0,1) \to \mathbb{R}$ is integrable?

   A measurable function $f : (0,1) \to \mathbb{R}$ is said to be *mean-square integrable* if $\int_{(0,1)} f^2 dm < \infty$.

   c. Show that every mean-square integrable function is integrable.

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *monotone convergence theorem.*

   b. Suppose $f : \mathbb{R} \to [0, \infty)$ is a measurable function. Show that

   $$\lim_{n\to\infty} \int_{\mathbb{R}} n \ln\left(1 + \frac{f}{n}\right) dm = \int_{\mathbb{R}} f dm.$$

   (Hint: Recall that $(1 + x/n)^n$ increases to e^x as $n \to \infty$ if $x \geq 0$.)

   c. State the *dominated convergence theorem.*

   d. Evaluate

   $$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n \sin(x/n)}{x(x^2+1)} dx$$

   (and justify the result).

4. Consider the (Lebesgue) measurable function $f : \mathbb{R}^2 \to \mathbb{R}$.
   a. What does *Fubini's theorem* tell us about $\int_{\mathbb{R}^2} f dm_2$?
   b. Show that if f is the joint density function of the absolutely continuous random variables X and Y, then X and Y are independent if and only if
   $$f(x, y) = f_X(x) f_Y(y) \text{ a.e.}$$

5. Let $\mu_i,\ i = 1, 2, 3$ be finite measures on a measurable space $(\Omega, \mathcal{F})$.
   a. What is meant by $\mu_1 \ll \mu_2$ (μ_1 is *absolutely continuous with respect to* μ_2)?
   b. What does the *Radon-Nikodym Theorem* say about the relation between μ_1 and μ_2 if $\mu_1 \ll \mu_2$?
   c. Let $\mu_1 = \delta_0 + \delta_1, \mu_2 = m_{[0,1]}$ and $\mu_3 = \mu_1 + \mu_2$. For which $i \neq j$ do we have $\mu_i \ll \mu_j$? Find the *Radon-Nikodym derivative* in each such case.

6. Consider the probability space $([0, 1], \mathcal{M}_{[0,1]}, m_{[0,1]})$ and, for $n = 1, 2, \ldots$, set
   $$X_n(\omega) = \begin{cases} n^{2/3} & \text{if } 0 \leq \omega < \frac{1}{n} \\ n^{-1/3} & \text{if } \frac{1}{n} \leq \omega \leq 1. \end{cases}$$
   a. Determine the distribution function $F_n(x)$ of X_n and utilize it to show that X_n converges weakly to 0.
   b. Which of the following statements are true? (Justify your answers).
      (i) $X_n \to 0$ in probability.
      (ii) $X_n \to 0$ almost surely.
      (iii) $X_n \to 0$ pointwise.
      (iv) $X_n \to 0$ in L^1-norm.
      (v) $X_n \to 0$ in L^2-norm.
      (vi) $X_n \to 0$ uniformly.

1				2			3				4		5			6		Σ
a	b	c	d	a	b	c	a	b	c	d	a	b	a	b	c	a	b	
2	2	2	4	2	2	3	2	3	2	3	2	3	2	2	3	3	3	45

Mark: $\frac{\text{Total}}{45} \times 9 + 1$ (rounded)