1 a $\Omega \in \mathcal{F}$

$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

2 $A_n \in \mathcal{F},\ n=1,2,\ldots \Rightarrow \bigcup_n A_n \in \mathcal{F}$

b $\mu(\emptyset) = 0$

$A \subset B \subset \Omega \Rightarrow \mu(A) \leq \mu(B)$

2 $A_n \subset \Omega,\ n=1,2\ldots \Rightarrow \mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$

c $\mathcal{F}$ is σ-field

2 $A_n \in \mathcal{F},\ n=1,2,\ldots$ disjoint $\Rightarrow \mu(\bigcup_n A_n) = \sum_n \mu(A_n)$

d Let $A_n \in \mathcal{F},\ n=1,2\ldots$ disjoint

$$E_i := \bigcup_{n=i}^{\infty} A_n \Rightarrow E_1 \supset E_2 \supset \ldots \text{ and } \bigcap_i E_i = \emptyset$$

$$E_1 = \bigcup_{n=1}^{i-1} A_n \cup E_i$$

4

$$\therefore \mu(E_1) = \mu(\bigcup_{n=1}^{i-1} A_n) + \mu(E_i)$$

$$= \sum_{n=1}^{i-1} \mu(A_n) + \mu(E_i)$$

$$\sum_{n=1}^{\infty} \mu(A_n) = \lim_{i\to\infty} \sum_{n=1}^{i-1} \mu(A_n)$$

$$= \lim_{i\to\infty} \{ \mu(E_1) - \mu(E_i) \}$$

$$= \mu(E_1) - \lim_{i\to\infty} \mu(E_i)$$

$$= \mu(\bigcup_n A_n)$$

so μ is measure

2 2 a $f^{-1}(I) \in \mathcal{M}_{(0,1)}$ for any interval I

b $\int_{(0,1)} |f| \, dm_{(0,1)} < \infty$

2 or $\int_{(0,1)} f^+ \, dm_{(0,1)} < \infty$ and $\int_{(0,1)} f^- \, dm_{(0,1)} < \infty$

c Let $A = \{ x \in (0,1) : f(x) > 1 \}$

$$\int_{(0,1)} |f| \, dm_{(0,1)} = \int_A |f| \, dm_{(0,1)} + \int_{A^c} |f| \, dm_{(0,1)}$$

3
$$\leq \int_A f^2 \, dm_{(0,1)} + \int_{A^c} 1 \, dm_{(0,1)}$$

$$\leq \int_{(0,1)} f^2 \, dm_{(0,1)} + 1 < \infty$$

so f is integ~~ra~~able

3 a if $f_n \geq 0$ measurable and $f_n \nearrow f$ pointwise, then

2
$$\lim_{n \to \infty} \int_E f_n \, dm = \int_E f \, dm$$

b Let $f_n = n \ln\left(1 + \frac{f}{n}\right) = \ln\left(1 + \frac{f}{n}\right)^n$

then $f_n \geq 0$ and f_n is measurable

3 (e.g. ~~Thm~~ Lemma 3.7)

Also $f_1 \leq f_2 \leq \ldots$ since $\ln$ is increasing and $(1 + f(x)/n)^n$ is increasing

MCT:

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \ln\left(1 + \frac{f(x)}{n}\right)^n = \ln e^{f(x)} = f(x)$$

so

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n \, dm = \int_{\mathbb{R}} f \, dm$$

3 c if $|f_n| \le g$ a.e. $f = \lim_{n\to\infty} f_n$ a.e. and $g \in \mathcal{L}^1(E)$, then $f \in \mathcal{L}^1(E)$ and

2
$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm$$

d
$$|f_n(x)| = \left| \frac{\sin(x/n)}{x/n} \frac{1}{x^2+1} \right| \le \frac{1}{x^2+1} =: g$$

$$\int_{\mathbb{R}} \frac{1}{1+x^2} \, dx = \arctan x \Big|_{-\infty}^{\infty} = \pi$$

3

DCT: $\displaystyle \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) \, dx = \int_{\mathbb{R}} \frac{1}{1+x^2} \, dx = \pi$

since $\displaystyle \lim_{n\to\infty} \frac{\sin(x/n)}{x/n} = 1$ (or MCT)

4 a if $f \in \mathcal{L}^1(\mathbb{R}^2)$ then

2
$$\int_{\mathbb{R}^2} f \, dm_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \, dx \, dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \, dy \, dx$$

b if $f(x,y) = f_X(x) f_Y(y)$ then

$$P_{(X,Y)}(B_1 \times B_2) = \int_{B_1 \times B_2} f(x,y) \, dm_2(x,y)$$

3
$$= \int_{B_1 \times B_2} f_X(x) f_Y(y) \, dm_2 = P_X(B_1) P_Y(B_2)$$

if $P_{(X,Y)}(B_1 \times B_2) = P_X(B_1) P_Y(B_2)$ then

$$\int_{B_1 \times B_2} f(x,y) \, dm_2(x,y) = P_{(X,Y)}(B_1 \times B_2) = P_X(B_1) P_X(B_2)$$

$$= \int_{B_1} \int_{B_2} f_X(x) f_Y(y) \, dm_2(x,y) \implies f(x,y) = f_X(x) f_Y(y)$$

5 a $\mu_1 \ll \mu_2 \iff \forall_{A\in\mathcal{F}} \left[\mu_2(A)=0 \Rightarrow \mu_1(A)=0\right]$

2

b if $\mu_1 \ll \mu_2$ then there exists a measurable function h, unique μ_2-a.e., such that for all $A \in \mathcal{F}$

2

$$\mu_1(A) = \int_A h \, d\mu_2 \qquad \left(h = \frac{d\mu_1}{d\mu_2}\right)$$

c $\mu_1 \ll \mu_3 \ , \ \mu_2 \ll \mu_3$

$$\mu_1(A) = \int_A \frac{d\mu_1}{d\mu_3}\, d\mu_3 = \int_A \frac{d\mu_1}{d\mu_3}\, d\mu_1 + \int_A \frac{d\mu_1}{d\mu_3}\, d\mu_2$$

$$\mathbb{1}_A(0) + \mathbb{1}_A(1) = \frac{d\mu_1}{d\mu_3}(0)\,\mathbb{1}_A(0) + \frac{d\mu_1}{d\mu_3}(1)\,\mathbb{1}_A(1) + \int_A \frac{d\mu_1}{d\mu_3}\, dm_{[0,1]}$$

3

$$\therefore \quad \frac{d\mu_1}{d\mu_3}(x) = \mathbb{1}_{\{0,1\}}(x)$$

$$\mu_2(A) = \int_A \frac{d\mu_2}{d\mu_3}\, d\mu_3 = \int_A \frac{d\mu_2}{d\mu_3}\, d\mu_1 + \int_A \frac{d\mu_2}{d\mu_3}\, d\mu_2$$

$$m_{[0,1]}(A) = \frac{d\mu_2}{d\mu_3}(0)\,\mathbb{1}_A(0) + \frac{d\mu_2}{d\mu_3}(1)\,\mathbb{1}_A(1) + \int_A \frac{d\mu_2}{d\mu_3}\, dm_{[0,1]}$$

$$\therefore \quad \frac{d\mu_2}{d\mu_3}(x) = \mathbb{1}_{[0,1]}(x)$$

6 a

$$F_n(x) = P(X_n \le x) = \begin{cases} 0 & x < n^{-1/3} \\ 1 - \frac{1}{n} & n^{-1/3} \le x < n^{2/3} \\ 1 & n^{2/3} \le x \end{cases}$$

3

$$EX_n = \int X_n \, dm_{[0,1]} = 2n^{\frac{2}{3}} \cdot \frac{1}{n} + n^{-\frac{1}{3}}\left(1 - \frac{1}{n}\right)$$

$$= 2n^{-1/3} - n^{-4/3}$$

$$F_n(x) \to F(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases}$$ except if $x=0$, $\Rightarrow$ weak conv.

6b. $X_n(0) = n^{2/3} \not\to 0$

3

so not pointwise, not uniformly

$X_n(\omega) \to 0 \qquad 0 < \omega \leq 1$

so almost surely, ~~and~~ and hence i.p. and weakly

$$\int_{[0,1]} |X_n|^p \, dm_{[0,1]} = \frac{1}{n} n^{2p/3} + \left(1 - \frac{1}{n}\right) n^{-p/3}$$

$$p=1: \quad 2n^{-1/3} - n^{-4/3} \to 0$$

$$p=2: \quad n^{1/3} + n^{-2/3} - n^{-5/3} \not\to 0$$