def 2.3, p. 20 — 1 a. $m^*(A) = \inf Z_A$

where

$$Z_A = \{ \sum_{n=1}^{\infty} \ell(I_n) \mid I_n \text{ interval}, A \subset \bigcup_n I_n \}$$

def 2.9, p. 27 — b. if, for all $B \subset \mathbb{R}$,

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c) \quad (*)$$

Ex. 2.4, p. 26 — c. m^* is subadditive and hence

$$m^*(B \cap A) \leq m^*(A)$$
$$m^*(B \cap A^c) \leq m^*(B)$$
$$m^*(B) \leq m^*(B \cap A) + m^*(B \cap A^c)$$

so if $m^*(A) = 0$ then $(*)$ is satisfied.

Rem. 2.12, p. 29 — 2 a. if, for pairwise disjoint sets E_i,

$$\mu(\bigcup_i E_i) = \sum_i \mu(E_i)$$

def 2.32, p. 46 — b. if μ is a measure and $\mu(\Omega) = 1$

def 3.1, p. 57 — 3 a. if, for any interval I, $f^{-1}(I) \in \mathcal{M}$

thm 3.9, p. 64, $I = [a, \infty)$ — b. we may restrict ourselves to interval of the type $[a, \infty)$:

$$h^{-1}([a,\infty)) = \{x \mid f(x) \geq a \text{ and } g(x) \geq a\}$$
$$= \{x \mid f(x) \geq a\} \cap \{x \mid g(x) \geq a\}$$
$$= f^{-1}([a,\infty)) \cap g^{-1}([a,\infty)) \in \mathcal{M}$$

2

Thm 4.24 4. (i) $P_X(A) \geq 0$ since $\int_A X\,dP \geq 0$

(ii) $P_X(\Omega) = 1$

(iii) let $A_1, A_2 \dots$ pairwise disjoint

$Y_n = \sum_{i=1}^{n} X 1_{A_i}$ $\quad Y = X 1_{\bigcup_i A_i}$

then $Y_n \nearrow Y$ (easy)

hence

$$\int_{\bigcup_i A_i} X\,dP = \int X 1_{\bigcup_i A_i}\,dP = \int Y\,dP$$

$$= \lim_{n\to\infty} \int Y_n\,dP = \lim_{n\to\infty} \int \sum_{i=1}^{n} 1_{A_i} X\,dP$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \int X 1_{A_i}\,dP = \sum_{i=1}^{\infty} \int_{A_i} X\,dP$$

that is $P_X(\bigcup_i A_i) = \sum_{i=1}^{n} P_X(A_i)$

2

5 a $|f_n| \leq g$ a.e. g integrable over E
$f = \lim_{n\to\infty} f_n$ then f integrable and

$$\lim_{n\to\infty} \int_E f_n\,dm = \int_E f\,dm$$

b. $\left| \frac{n \sin x}{1+n^2 x^{1/2}} \right| \leq \frac{n}{1+n^2 x^{1/2}} \leq \frac{1}{n\sqrt{x}} \leq \frac{1}{\sqrt{x}} \in \mathcal{L}_{[0,1]}$

$\lim_{n\to\infty} \frac{n \sin x}{1+n^2 x^{1/2}} = 0$ DCT: $\lim_{n\to\infty} \int f_n\,dm = 0$

2

6 a $F_X(x) = m(\{\omega : X(\omega) \leq x\})$

$$= \begin{cases} m(\varnothing) = 0 & x < 0 \\ m(\bar{\mathbb{Q}} \cap [0,1]) = 1 & 0 \leq x < 1 \\ m([0,1]) = 1 & x \geq 1 \end{cases}$$

$EX = \int_{[0,1]} X \, dm = \int_{[0,1]} 0 \, dm = 0$ for $X = 0$ a.e.

b $F_X(x) = m(\{\omega \in [0,1] : \omega^2 \leq x\})$

$$= \begin{cases} 0 & x < 0 \\ \sqrt{x} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$f_X(x) = \frac{1}{2} x^{-\frac{1}{2}} 1_{(0,1)}(x)$

$EX = \int_{\mathbb{R}} x f_X(x) \, dx = \int_0^1 \frac{1}{2} \sqrt{x} \, dx = \frac{1}{3}$

3

7 a $\int_{\mathbb{R}} G_c(x) \, dm(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{(x,x+c]}(y) \, dP(y) \, dm(x)$

$= \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{(x,x+c]}(y) \, dm(x) \, dP(y) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[y-c,y)}(x) \, dm(x) \, dP(y)$

$= \int_{\mathbb{R}} c \, dP(y) = c$

b $\int_{\mathbb{R}} F(x) \, dP(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{(-\infty,x]}(y) \, dP(y) \, dP(x)$

$= \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[y,\infty)}(x) \, dP(x) \, dP(y) = \int_{\mathbb{R}} (1 - F(y-)) \, dP(y) = 1 - \int_{\mathbb{R}} F(y) \, dP(y)$

$\therefore \int_{\mathbb{R}} F(x) \, dP(x) = \frac{1}{2}$

2

8 $f_n(0) = n^2 \to \infty$ $(n \to \infty)$, so

a. not uniformly

b. not pointwise

c. $f_n(x) \to 0$ $(n \to \infty)$ $\forall x \neq 0$ so
$f_n \to f$ a.e.

d. $\int_{\mathbb{R}} |f_n|^p \, dm = 2\int_0^\infty n^{2p} e^{-npx} \, dx$

$= 2n^{2p} \frac{1}{np} = \frac{2}{p} n^{2p-1} \to 0 \iff p < \frac{1}{2}$