Reference: EWI08/TW/SK/011/EvD
Date: 15th January 2008

Exam Measure and Probability (157040)
Monday, 21 January 2008, 9.00 - 12.00 a.m.

This exam consists of 7 problems

1. Let Ω be a non-empty set.

   a. Define what is meant by a σ-*field* of subsets of Ω.

   b. Let $\mathcal{F}$ be a σ-field of subsets of Ω and let $B \subseteq \Omega$. Show that $\mathcal{G} = \{A \in \mathcal{F} : A \subseteq B \text{ or } B^c \subseteq A\}$ is a σ-field.

   c. Let $f : \Omega \to \mathbb{R}$ be a function and let $\mathcal{F}$ be a σ-field. What does it mean to say that f is $\mathcal{F}$-*measurable*.

   d. Let $\mathcal{F}, \mathcal{G}$ be as in b, and let $f : \Omega \to \mathbb{R}$ be an $\mathcal{F}$-measurable function. Under what conditions will f be $\mathcal{G}$-measurable?

2. Let Ω be a non-empty set, $\mathcal{F}$ a σ-field of subsets of Ω, and μ a $[-\infty, \infty]$-valued function on $\mathcal{F}$. When do we call μ

   a. an *outer measure*?

   b. a *measure*?

   c. a *probability measure*?

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *dominated convergence theorem*.

   b. Investigate the convergence as $n \to \infty$ of
   $$\int_0^\infty \frac{1}{(1+x^2)\sqrt[n]{x}}dx.$$

4. Consider the probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$. Find F_X, the distribution function, and $\mathbb{E}(X)$, the expectation of

   a. the random variable X, given by $X(\omega) = 3\omega - 2$,

   b. the random variable X given by $X(\omega) = \min(\omega, 1-\omega)$.

5. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces, and let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$.

a. Under which condition do we have

$$\int_{\Omega_1}\left(\int_{\Omega_2} f d\mu_2\right) d\mu_1 = \int_{\Omega_2}\left(\int_{\Omega_1} f d\mu_1\right) d\mu_2?$$

(This is *Fubini's Theorem*.)

b. Use Fubini's Theorem to show that for any distribution function F

$$\int_{\mathbb{R}} (F(x+a) - F(x))dx = a.$$

6. Let $(\Omega, \mathcal{F})$ be a measurable space and let μ and ν be finite measures on $(\Omega, \mathcal{F})$.

   a. What is meant by $\nu \ll \mu$ (ν is *absolutely continuous with respect to* μ)?

   b. What does the *Radon-Nikodym Theorem* say about the relation between μ and ν if $\nu \ll \mu$?

   c. Give the *Radon-Nikodym derivative* $\frac{d\nu}{d\mu}$ if $\nu \ll \mu$ and Ω is finite.

7. Consider the probability space $([0,1), \mathcal{M}_{[0,1)}, m_{[0,1)})$ and, for $n = 1, 2, \ldots$, set

$$X_n(\omega) = \begin{cases} 0 & \text{if } \; 0 \le \omega < \frac{1}{2} - \frac{1}{2n} \\ n & \text{if } \; \frac{1}{2} - \frac{1}{2n} \le \omega < \frac{1}{2} \\ \frac{1}{n} & \text{if } \; \frac{1}{2} \le \omega < 1. \end{cases}$$

   a. Find the distribution function $F_n(x)$ of X_n.

   b. Which of the following statements are true? (Justify your answers).

      (i) $X_n \to 0$ in probability.

      (ii) $X_n \to 0$ weakly.

      (iii) $X_n \to 0$ almost surely.

      (iv) $X_n \to 0$ pointwise.

      (v) $X_n \to 0$ in L^1-norm.

      (vi) $X_n \to 0$ in L^2-norm.

      (vii) $X_n \to 0$ uniformly.

points:

1	2	3	4	5	6	7
7	5	4	4	5	6	5

mark: $\frac{\text{total}}{36} \times 9 + 1$ (rounded)