Kenmerk: EWI/TW/SP/10-063
Datum: 21 december 2010

Exam Measure and Probability (191570401)
Monday 17 January, 8.45 - 11.45 a.m.

This exam consists of 7 problems

1. a. Define what is meant by $m^*(A)$, the Lebesgue outer measure of $A \subset \mathbb{R}$.

   b. Use the countable subadditivity (and the definition) of Lebesgue outer measure to show that $m^*(A) = 0$ implies $m^*(A \cup B) = m^*(B)$ for each $B \subset \mathbb{R}$.

   c. Define what is meant by saying that $A \subset \mathbb{R}$ is (Lebesgue) measurable.

2. Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mu : \mathcal{F} \to [0, \infty)$ be a *finitely-additive* set function, that is, a function such that $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever A and B are disjoint sets in $\mathcal{F}$. Suppose μ has the following property: If $E_1 \supset E_2 \supset E_3 \supset \dots$ are sets in $\mathcal{F}$ such that $\cap_i E_i = \emptyset$, then $\lim_{i\to\infty} \mu(E_i) = 0$. Prove that μ is a measure on $(\Omega, \mathcal{F})$.

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.

   a. State the *dominated convergence theorem.*

   b. Evaluate
   $$\lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2x^2}dx$$
   for all $a \in \mathbb{R}$ (and justify the result).

4. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces, and let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$.

   a. State the condition in *Fubini's theorem* under which we have
   $$\int_{\Omega_1\times\Omega_2} fd(\mu_1 \times \mu_2) = \int_{\Omega_1} \left(\int_{\Omega_2} fd\mu_2\right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} fd\mu_1\right) d\mu_2.$$

   b. Use Fubini's Theorem to show that for any (probability) distribution function F
   $$\int_{\mathbb{R}} (F(x+a) - F(x))dx = a.$$

5. Let X and Y be two random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with joint density

$$f_{X,Y}(x, y) = \mathbf{1}_A(x, y), \quad (x, y) \in \mathbb{R}^2,$$

where A is the triangle with corners at (0,2), (1,0) and (1,2).

a. Find the conditional density $f_{X|Y}(x|Y = y)$ of X given $Y = y$.

b. Determine $E(X|Y)$.

6. Let μ and ν be two finite measures on a measurable space $(\Omega, \mathcal{F})$, and suppose that, for some $a > 0$, $b > 0$, we have $a\mu(A) \leq \nu(A) \leq b\mu(A)$ for all $A \in \mathcal{F}$.

a. Show that μ and ν are equivalent measures (that is, $\mu \ll \nu$ and $\nu \ll \mu$).

b. Using the fact that $f \leq g$ μ-a.e. if $\int_A f d\mu \leq \int_A g d\mu$ for all $A \in \mathcal{F}$, show that the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ satisfies $a \leq \frac{d\nu}{d\mu} \leq b$ μ-a.e.

7. Consider the probability space $([0, 1], \mathcal{M}_{[0,1]}, m_{[0,1]})$ and, for $n = 1, 2, \ldots$, set

$$X_n(\omega) = \begin{cases} \frac{n}{\log n} & \text{if } \ 0 \leq \omega < \frac{1}{n} \\ 0 & \text{if } \ \frac{1}{n} \leq \omega \leq 1. \end{cases}$$

a. Find the distribution function $F_n(x)$ of X_n and $\mathbb{E}(X_n)$.

b. Which of the following statements are true? (Justify your answers).

(i) $X_n \to 0$ in probability.

(ii) $X_n \to 0$ almost surely.

(iii) $X_n \to 0$ pointwise.

(iv) $X_n \to 0$ in L^1-norm.

(v) $X_n \to 0$ in L^2-norm.

(vi) $X_n \to 0$ uniformly.

1	2	3	4	5	6	7
4	3	3	3	3	3	4

Mark: $\frac{\text{Total}}{23} \times 9 + 1$ (rounded)