Kenmerk: EWI/TW/SP/12-001
Datum: 16 januari 2012

Exam Measure and Probability (191570401)
Monday 23 January 2012, 8.45 - 11.45 a.m.

This exam consists of 7 problems

1. Let Ω be a set, $\mathcal{F}$ a collection of subsets of Ω and $\mu : \mathcal{F} \to \mathbb{R}$. When do we call
   a. $\mathcal{F}$ a σ-field?
   b. μ an outer measure?
   c. μ a measure?
   d. $(\Omega, \mathcal{F}, \mu)$ a probability space?

2. Consider the measure space $((0,1), \mathcal{M}_{(0,1)}, m_{(0,1)})$.
   a. Define what is meant by saying that $f : (0,1) \to \mathbb{R}$ is measurable.
   b. Define what is meant by saying that $f : (0,1) \to \mathbb{R}$ is integrable.

   A measurable function $f : (0,1) \to \mathbb{R}$ is said to be *mean-square integrable* if $\int_{(0,1)} f^2 dm < \infty$.

   c. Show that every mean-square integrable function is integrable.
   (Hint: Separate the cases $|f| \geq 1$ and $|f| < 1$.)

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.
   a. State the *monotone convergence theorem.*
   b. (*Borel-Cantelli lemma*) Suppose $\{E_k\}$ is a sequence of measurable sets satisfying
   $$\sum_{k=1}^{\infty} m(E_k) < \infty.$$
   Show that $m(F) = 0$ when $F = \{x : x \text{ belongs to infinitely many sets } E_k\}$.
   (Hint: Define $f_n = \sum_{k=1}^{n} \mathbb{I}_{E_k}$, $f = \lim_{n\to\infty} f_n$, and show that $\int_{\mathbb{R}} f dm < \infty$.)

4. Consider the measure space $(\mathbb{R}, \mathcal{M}, m)$.
   a. State the *dominated convergence theorem.*
   b. Evaluate
   $$\lim_{n\to\infty} \int_0^{\infty} \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx.$$
   (Hint: Recall that $(1 + x/n)^n$ increases to e^x as $n \to \infty$.)

5. Consider the (Lebesgue) measurable function $f : \mathbb{R}^2 \to \mathbb{R}$.

   a. What does *Fubini's theorem* tell us about $\iint_{\mathbb{R}^2} f dm_2$?

   b. Show that if f is the joint density function of the absolutely continuous random variables X and Y, then X and Y are independent if and only if

$$f(x,y) = f_X(x) f_Y(y) \text{ a.e.}$$

6. Consider the interval $[-1,1]$ with Lebesgue measure $m_{[-1,1]}$. and let ν be a measure on the measurable space $([-1,1], \mathcal{B}_{[-1,1]})$ such that

$$\nu([-1,x]) = \begin{cases} 0 & \text{if } -1 \le x < 0 \\ 1 + x^2 & \text{if } 0 \le x \le 1. \end{cases}$$

   a. Show that ν is *not* absolutely continuous with respect to $m_{[-1,1]}$.

   b. Give the Lebesgue decomposition of ν with respect to $m_{[-1,1]}$, that is, determine ν_a and ν_s such that $\nu = \nu_a + \nu_s$, $\nu_a \ll m_{[-1,1]}$ and $\nu_s \perp m_{[-1,1]}$.

   c. Determine the Radon-Nikodym derivative of ν_a with respect to $m_{[-1,1]}$.

7. Consider the probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$ and, for $n = 1, 2, \ldots,$ set

$$X_n(\omega) = \begin{cases} \frac{n}{\log n} & \text{if } 0 \le \omega < \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \le \omega \le 1. \end{cases}$$

   a. Find the distribution function $F_n(x)$ of X_n and $\mathbb{E}(X_n)$.

   b. Which of the following statements are true? (Justify your answers).

      (i) $X_n \to 0$ in probability.

      (ii) $X_n \to 0$ almost surely.

      (iii) $X_n \to 0$ pointwise.

      (iv) $X_n \to 0$ in L^1-norm.

      (v) $X_n \to 0$ in L^2-norm.

Normering:

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Mark: $\frac{\text{Total}}{29} \times 9 + 1$ (rounded)