Kenmerk: EWI2015/dmmp/005/BM

Exam Optimization Modeling (191581420)

Friday, April 17, 2015, 8:45 – 11:45

- Use of calculators, mobile phones, etc. is not allowed!
- This exam consists of four problems. Start a new page for every problem.
- Total number of points: 45 + 5 = 50. Distribution of points according to the following table.

1: 12	2a: 2	3: 12	4a: 2
	2b: 2		4b: 3
	2c: 1		4c: 3
	2d: 1		4d: 3
	2e: 4		

1. Vehicle Routing and Traveling Salesman Problem

You work for a logistics company, which has k trucks in city s. Your task is to assign routes to the k trucks such that the following conditions are satisfied:

(i) Every truck starts in city s. Each of the other $n-1$ cities has to be visited exactly once by exactly one truck.

(ii) After visiting their cities, the trucks return to s. They do not return to s between any two of the cities assigned to them.

(iii) There is a set P of pairs of cities with the following interpretation: If $(u, v) \in P$, then u has to be visited by the same truck as v, and u has to be visited before v. (The truck can visit cities after visiting u and before visiting v.)

Your goal is to minimize the total costs of the company. Costs arise as follows:

(v) There is a known cost $c_{u,v}$ of going from city u to city v for all u and v. (We can have $c_{u,v} \neq c_{v,u}$.)

(vi) There is a fixed cost t for using a truck. (You do not have to use all trucks.)

(12 points) Build an ILP model for this problem. If you use multiplication of binary variables, you do not have to linearize it. For the rest, your model has to be linear.

2. Knapsack and Column Generation

We consider the following variant of the knapsack problem: we are given items $1, \ldots, n$ and a weight bound B. Item i has a weight of w_i and yields a profit of p_i. There are infinitely many copies of each item. Formulated as an IP, we have

$$\text{maximize} \sum_{i=1}^{n} p_i x_i$$
$$\text{subject to} \sum_{i=1}^{n} w_i x_i \leq B \text{ and } x_1, \ldots, x_n \in \mathbb{N}.$$

We consider the relaxation

$$\begin{aligned} \text{maximize} & \sum_{i=1}^{n} p_i x_i \\ \text{subject to} & \sum_{i=1}^{n} w_i x_i \leq B \text{ and} \\ & x_1, \ldots, x_n \geq 0. \end{aligned} \tag{KS}$$

(a) *(2 points)* Give an optimal solution of the relaxation (KS).

Hint: You can assume that the items are sorted in a way that is convenient for you.

(b) *(2 points)* Write down the dual of (KS).

We want to solve (KS) with column generation. Thus, assume that we have a set $S \subseteq \{1, \ldots, n\}$ of items whose variables we use. In every step, we want to add the item whose variable has the smallest reduced costs, if there is a variable with negative reduced costs.

(c) *(1 point)* What is an optimal solution of (KS) restricted to items in the set S? The LP restricted to variables with index in S is given by

$$\text{maximize} \sum_{i \in S} p_i x_i$$
$$\text{subject to} \sum_{i \in S} w_i x_i \leq B \text{ and } x_i \geq 0 \text{ for } i \in S.$$

(d) *(1 point)* Given your optimal solution from Part (c), what is the corresponding dual solution?

Hint: complementary slackness.

(e) *(4 points)* Given your solution of Part (c) and the corresponding dual solution of Part (d), what are the reduced costs of variables $x_1, \ldots, x_n$?

Which variable should be added to the basis?

3. Store Location

You want to open a couple of branches of a coffee shop in a city. Of course, it is bad to open two branches very close to each other, as then they take away customers from each other.

Market research shows the following:

- Between any two possible locations p and q, there is a "concurrency parameter" $c(p,q)$. It has turned out that $c(p,q) = c(q,p) \geq 0$ for all $p \neq q$. The larger $c(p,q)$, the fiercer the competition between p and q.
- Every location p comes with a potential gain $g(p)$.
- If you have opened the n branches at locations $p_1, \ldots, p_n$, then the total gain is

$$\frac{\sum_{i=1}^{n} g(p_i)}{1 + \sum_{1 \leq i < j \leq n}^{\ell} c(p_i, p_j)}.$$

- The potential gains $g(p)$ of the locations p, the parameters $c(p,q)$, and the number n of branches that you want to open are given.

(12 points) Build an ILP model that maximizes your total gain. (Apply modeling tricks explicitely to make your model linear where applicable.)

4. Miscellaneous Questions

(a) *(2 points)* Consider the LP

$$\begin{aligned} &\text{minimize } y \\ &\text{subject to } x \leq 2, \\ &\qquad\qquad\quad x, y \geq 0. \end{aligned}$$

Assume that you run the simplex method to solve this problem. Is it possible that simplex outputs the solution $x = 1$, $y = 0$? Why or why not?

(b) *(3 points)* Consider the integer program

$$\begin{aligned} \text{minimize } & c^{\mathrm{T}} x \\ \text{subject to } & Ax = b, \\ & x \in \mathbb{N}^d. \end{aligned}$$

Let z_{IP} be the objective value of an optimum solution, and let z_{LP} be the objective value of the relaxation of this IP, i.e., when we replace $x \in \mathbb{N}^d$ by $x \geq 0$. (You can assume that both z_{IP} and z_{LP} exist, i.e., both LP and IP are neither unbounded nor infeasible.)

Which of the following cases can occur?

(i) $z_{\text{IP}} < z_{\text{LP}}$?

(ii) $z_{\text{IP}} = z_{\text{LP}}$?

(iii) $z_{\text{IP}} > z_{\text{LP}}$?

(c) *(3 points)* Consider again the IP of Question (b). Assume that $b \in \mathbb{Z}^n$ and that A is a totally unimodular.

Which of the following cases can occur?

(i) $z_{\text{IP}} < z_{\text{LP}}$?

(ii) $z_{\text{IP}} = z_{\text{LP}}$?

(iii) $z_{\text{IP}} > z_{\text{LP}}$?

(d) *(3 points)* Prove or disprove the following statement.

For *every* matrix $A \in \mathbb{Z}^{n \times d}$, there *exists* a vector $b \in \mathbb{Z}^n$ with $b \neq 0$ such that the LP

$$\begin{aligned} \text{minimize } & c^{\mathrm{T}} x \\ \text{subject to } & Ax = b, \\ & x \geq 0 \end{aligned}$$

has only integral basic feasible solutions.