Exam Optimization Modeling (191581420)

Thursday, April 18, 2019, 8:45 – 11:45

- Use of calculators, mobile phones, etc. is not allowed!
- This exam consists of four problems. Start a new page for every problem.
- Total number of points: 30. The distribution of points is according to the following table.

1a: 2	2: 8	3: 8	4a: 2
1b: 2			4b: 4
1c: 2			
1d: 2			

1. Modeling Tricks

(a) *(2 points)* Assume that you are given parameters $a_1, \ldots, a_k \in \mathbb{R}$, an integer variable $x \in \{1, \ldots, k\}$, and an unrestricted variable y.

Model the constraint $y = a_x$.

(b) *(2 points)* You are given two constraints $a_1 x \leq b_1$ and $a_2 x \leq b_2$ with $a_1, a_2 \in \mathbb{Z}^d$, $b_1, b_2 \in \mathbb{Z}$, and $x \in \mathbb{Z}^d$ is a vector of binary variables.

Model the "exclusive or" of these two constraints, i.e., the following constraint: Either ($a_1 x \leq b_1$ and $a_2 x > b_2$) or ($a_1 x > b_1$ and $a_2 x \leq b_2$).

(c) *(2 points)* Given an integer variable $x \in \{2, \ldots, M\}$ for some large $M \in \mathbb{N}$, model the following constraint: "The value of x is not a prime number."

(d) *(2 points)* Model the constraint "$x \neq y$" for two integer variables x, y with range $\{0, \ldots, M\}$ using linear constraints (you are allowed to use additional variables).

2. Correlation Clustering

(8 points) We want to cluster a set X of points in a very particular way. We have a similarity measure s that measures how similar two points are: $s(x, y) = s(y, x) \in \mathbb{R}$ for all $x, y \in \mathbb{R}$. Note that both positive and negative values are allowed.

Our goal is to find a clustering, i.e., a partitioning of X into k sets $C_1, \ldots, C_k$ that maximizes that sum of pairwise similarity of all points in the same cluster, summed over all clusters. Note that the number k of clusters is not given or fixed, but arbitrary.

Example: If $s(x, y) = -1$ for all $x, y \in X$, then an optimal clustering consists of $|X|$ singleton clusters. If $s(x, y) = 1$ for all $x, y \in X$, then an optimal clustering consists of a single cluster that contains all points.

To avoid clusters that are too small, we are also given a number $\ell \in \mathbb{N}$. Only clusterings with $|C_i| \geq \ell$ for all $i \in \{1, \ldots, k\}$ are allowed. (This is not taken into account in the example above.)

Model this problem as an integer linear program.

3. Vehicle Routing with Precedence Constraints

(8 points) We are given k service cars, all located at a central depot. These service cars have to visit a set of clients. In order to build a schedule for the service cars, you have to take into account the following issues:

- Each client is visited by exactly one service car.
- There are precedence constraints between certain clients of the form "client i must be visited before client j". This does not mean that i must be visited directly before j, but both have to be visited by the same car, and this car has to visit i at some point in time and j at some later point in time.
- The service cars all return to the depot after their tours.
- Travel times between any pair of clients and to/from the depot are known.
- The goal is that the latest service car returns to the depot as early as possible.
- All cars are used.

Model this problem as an integer linear program.

4. Miscellaneous Questions

(a) *(2 points)* Prove or disprove the following statement: For *every* matrix $A \in \mathbb{Z}^{n \times d}$ with $\operatorname{rank}(A) = n$, there *exists* a vector $b \in \mathbb{Z}^n$ with $b \neq 0$ such that the LP

$$\begin{aligned} \text{minimize } & c^{\mathrm{T}} x \\ \text{subject to } & Ax = b, \\ & x \geq 0 \end{aligned}$$

has only integral basic feasible solutions.

Hint: Cramer's rule.

(b) *(4 points)* To model the TSP, let $G = (V, E)$ be an undirected, complete graph with $|V| = n$. We have binary variables x_e for $e \in E$ together with the degree constraint

$$\sum_{u \in V \setminus \{v\}} x_{\{u,v\}} = 2 \quad \text{for all } v \in V.$$

To avoid subtours, we are given two alternatives:

(i) For all cycles $v_0, v_1, v_2, \ldots, v_k = v_0$ with $k \in \{3, 4, \ldots, n-2\}$, we require

$$\sum_{i=0}^{k-1} x_{\{v_i, v_{i+1}\}} \leq k - 1.$$

(ii) For all subsets $S \subseteq V$ with $\emptyset \neq S \neq V$, we require

$$\sum_{e \in \delta(S)} x_e \geq 2.$$

Prove that these two constraints are equivalent: Every integer solution satisfies either none of them or both.