Exam: Optimization Modelling

Code : 158142
Date : 13:30-16:30, January 24, 2008
Preliminaries:
1. You may answer in Dutch.
2. Use of calculators, mobile phones, etc. are not allowed.
3. Every answer must be motivated!

1 Absolute values

Consider the following model statement:

$$\begin{aligned} \text{minimize } & \sum_j c_j |x_j| \\ \text{subject to : } & \sum_j a_{ij} x_j \leq b_i \quad \forall i \\ & x_j \text{ free} \end{aligned}$$

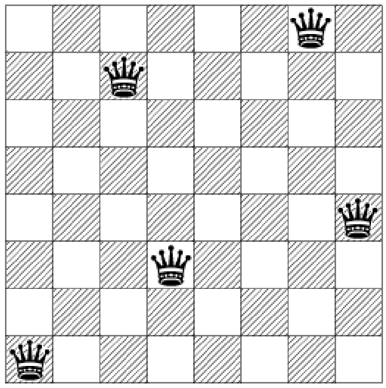
Assume that $c_j > 0$ for all j. Due to the absolute value in the objective it is not possible to directly apply linear programming. Formulate a linear programming model that is equivalent to the above. Provide a short argument why your linear model is equivalent to the above, and why the assumption $c_j > 0$ is needed.

2 Elimination of products of variables

Consider a model formulation with a product the of variables x_1 and x_2, i.e. the term $x_1 x_2$ occurs either in the constraints or the objective function.

1. Suppose x_1 and x_2 are both binary variables. How should we change the model formulation such that it becomes a linear model? Provide a short argument.
2. Suppose x_1 is a binary and x_2 is a continuous variable. What assumption has to be made and how to change the model formulation such that it becomes a linear model?

3 The Queen Domination Problem



The *Queen Domination Problem* is the problem of putting as many chess queens on an $n \times n$ chessboard as possible such that none of them is able to capture any other using the standard chess queen's moves. The queens must be placed in such a way that no two queens would be able to attack each other. Thus, a solution requires that no two queens share the same row, column, or diagonal. In the picture a solution is given with five queens on a standard sized chessboard.

Although this problem is notoriously hard to solve, it is not too difficult to formulate it as a optimization model. Build an optimization model for the Queen Domination Problem.

4 Multi-item lot sizing

Consider the *Multi-item lot sizing* problem which we discussed in one of the lectures. You can find the problem description and the basic linear programming formulation in Appendix A. (Tip: In the answers you give, you may refer to the constraints by their number.)

1. Argue that it is not necessary to require the variables y_{pt}, z_{pqt} and s_{pt} to be integer, and still have a valid model formulation. Why is this observation useful to us?

2. Which constraints can be dropped if we assume all coefficients of the objective to be positive? Give a short argument why they can be dropped and why this can be useful to us.

3. **Multiple machines**
   In the basic problem there was only one machine to be scheduled. Lets suppose that there are multiple machines available for production, and each machine can producing all types of product. However, not each machine is equally suitable for each product, this is reflected in the production rate, the minimum run and down times, and all costs (except storage costs). Write down an optimization model that can handle this multi-machine case.
   (Tip: For your and my convenience, base your model/notation on the one in the appendix, and make use of the observations in question 4.1 and 4.2)

4. **Changeover times**
   Up to now we have assumed that the changeover times where negligible. Lets suppose they are not. For each machine there is a changeover time of one day.
   You are asked to incorporate this in the multi-machine case of question 4.3.
   (Tip: Perhaps it is useful to read question 4.5 before answering this question.)

5. **Changeover crew**
   The actual change the production on a machine is performed by a crew of technicians. There is only one crew available, this implies that there can be no two machines in *changeover* at the same time.
   Incorporate this in the model you made for question 4.4.

5 Positioning Ambulances

In the guest lecture Dieuwke Vijselaar talked about *locating ambulances*. To remind you; the first problem she presented was the following.
To guarantee that a quick response to an emergency there should always be an ambulance within a 13 minute drive from each possible location (location being a zip-code area). There is only a small number of locations suitable for positioning an ambulance. From these locations a number must be selected to actually place an ambulance such that all locations are covered. Covered means that they are within a 13 minute drive from an ambulance. The objective is to use as few as possible ambulances.

1. Describe this coverage constraint.

2. At the guest lecture it was argued that the coverage constraint is bad for the performance of the optimization model since it is huge. Each zip-code is compared with all possible ambulance locations, which is in this instance 70.000 times 20 comparisons. How can this be reduced?

Determining your mark

In total you can earn up to 36 points, and you get 4 for free. The total points earned divided by 4 gives your mark. Point for each question are:

Question	1	2.1	2.2	3	4.1	4.2	4.3	4.4	4.5	5.1	5.2
Points	4	3	3	6	2	2	4	4	4	2	2

Appendix A: Multi-item lot sizing

Problem description
A set of products has to be produced on a single production line during a planning horizon. This horizon is divided into days.
Production takes place in the form of batches covering one or more days. The length of each batch is not known a priori, but should be determined by an optimization model.
For each product there is a minimum batch size in terms of days. In addition, there must be a minimum number of days between successive batches of the same product. There is a known demand for each product, and production during any particular day can also be used to satisfy the demand for that day. The production rate (quantity per day) for each product is known, and does not change over time.
There is a changeover cost when one batch is ended and a new one is started (due to equipment changes). The actual changing of the equipment needs very little time, and does not affect the level of production of either the ending batch or the starting batch.
Total cost over the planning horizon is determined by the cost to produce the products, to store the products, to start a batch, and to clean production equipment between particular changeovers (the changeover costs).

Basic linear programming formulation

Sets:

p, q	products
t, k	days

Parameters:

α_p	minimum run-time of a batch for product p
β_p	minimum down-time for product p
ρ_p	production rate, products p per day
d_{pt}	demand on day t for product p
e_{pq}	changeover cost when passing from the production of p to q
f_p	production cost for one day producing product p
g_p	start up cost for product p
h_p	storage cost per day per unit of product p

Variables:

x_{pt}	binary, product p is produced on day t
y_{pt}	binary, a batch of product p starts on day t
z_{pqt}	binary, product p is produced on day $t-1$ and q on day t
s_{pt}	integer, stock level of product p at the end of day t

Constraints:

$$x_{pt} - x_{p,t-1} \leq y_{pt} \qquad \forall p, t \geq 2 \tag{1}$$

$$x_{pt} = y_{pt} \qquad \forall p, t = 1 \tag{2}$$

$$y_{pt} \leq x_{pt} \qquad \forall p,t \tag{3}$$
$$y_{tp} \leq 1 - x_{p,t-1} \qquad \forall p, t \geq 2 \tag{4}$$
$$x_{p,t-1} + x_{q,t} - 1 \leq z_{pqt} \qquad \forall p,q,t \geq 2,\ with\ p \neq q \tag{5}$$
$$z_{pqt} \leq x_{p,t-1} \qquad \forall p,q,t \geq 2,\ with\ p \neq q \tag{6}$$
$$z_{pqt} \leq x_{q,t} \qquad \forall p,q,t,\ with\ p \neq q \tag{7}$$
$$y_{pt} \leq x_{p,k} \quad \forall p,t,k,\ with\ t \leq k \leq t + \alpha_p - 1 \tag{8}$$
$$y_{pk} \leq 1 - x_{p,t} \quad \forall p,t,k,\ with\ t+1 \leq k \leq t + \beta_p \tag{9}$$
$$\sum_p x_{pt} = 1 \qquad \forall t \tag{10}$$
$$s_{pt} = s_{p,t-1} + \rho x_{pt} - d_{pt} \qquad \forall p, t \geq 2 \tag{11}$$
$$x_{pt}, y_{pt} \in \{0,1\} \qquad \forall p,t$$
$$z_{pqt} \in \{0,1\} \qquad \forall p,q,t, with\ p \neq q$$
$$s_{pt} \geq 0,\ integer, \qquad \forall p,t$$

Objective:

$$\text{minimize}: \sum_t \sum_p \left(\sum_{q:q\neq p} e_{pq} z_{pqt} + f_p x_{pt} + g_p y_{pt} + h_p s_{pt} \right)$$

Constraints (1)-(4) connect the start up variables with the production variables. Constraint (5)-(7) connect the changeover variables with the production variables. Constraints (8)-(9) control the run and down time of a batch. Constraint (10) states that we produce only 1 type of product a day. Finally, constraint (11) takes care of the demand and stock level.