Kenmerk: EWI2013/dmmp/008/BM

Exam Optimization Modeling (191581420)

Tuesday, April 16, 2013, 8:45 – 11:45

- Use of calculators, mobile phones, etc. is not allowed!
- This exam consists of three problems. Please start a new page for every problem.
- Total number of points: 36 + 4 = 40. Distribution of points according to the following table.

1a: 8	2a: 8	3a: 3
1b: 8	2b: 3	3b: 3
		3c: 3

1. Clustering

We want to cluster a set X of points. More precisely, given a number k, we want to partition the points in X into subsets $C_1, \dots, C_k$. This means that $\bigcup_{i=1}^{k} C_i = X$ and $C_i \cap C_j = \emptyset$ for all $i \neq j$.

(a) Our goal is to minimize the sum of the distances of points within the same cluster, namely the function

$$\sum_{i=1}^{k} \sum_{x,y \in C_i} d(x,y),$$

where the distance between two points $x, y \in X$ is given by $d(x,y)$. We have $d(x,x) = 0$ and $d(x,y) \geq 0$ for all $x, y \in X$.

Model this problem as an ILP.

(b) Now we consider the following variant of the problem: The points X are on the real line, i.e., $X \subseteq [0, M]$ for some fixed large number M. In addition to the clusters $C_1, \dots, C_k$, we are looking for k points $z_1, \dots, z_k$ (called *representatives*) such that

$$\sum_{i=1}^{k} \sum_{x \in C_i} |x - z_i|$$

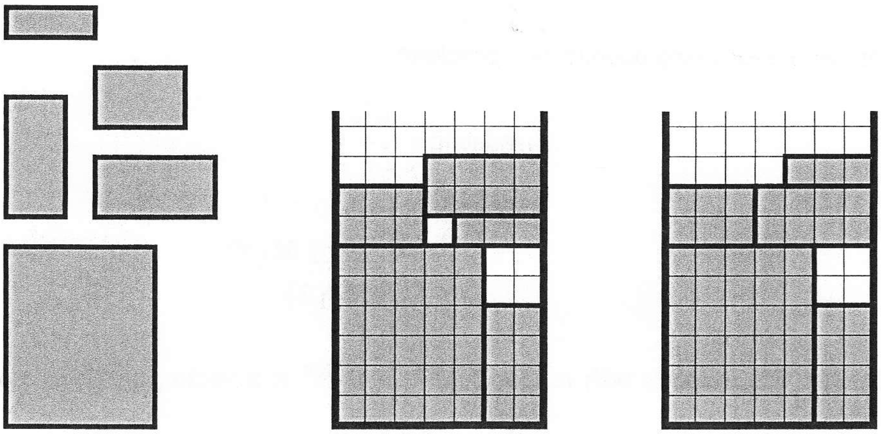
is minimized. This means that we want to minimize the sum over all clusters of the distances of the points in the cluster to the respective representative.

Model this problem as an ILP.

2. Simplified Tetris

In a simplified offline-version of Tetris, you are given a number of rectangular blocks, and your goal is to fill completely as many lines of the playing field as possible. (Note that, different from real Tetris, there is no ordering of blocks, i.e., there is nothing like "block a comes after block b and, thus, a cannot be underneath b".)

The following graph shows five blocks on the left-hand side and two solutions of a playing field of width $L = 8$ and these five blocks. The solution in the middle completely fills five lines. It is not optimal. The solution on the right-hand side for the same instance is optimal and fills six lines.

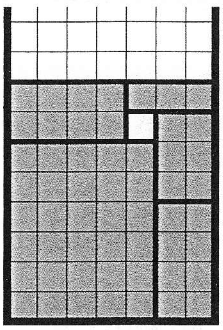


(a) Model the problem of completely filling as many rows as possible as an integer linear program. Rotation of blocks is not allowed. Note that, unlike in real Tetris, only rectangular blocks are allowed.

Hints: You might want to use a simple/trivial upper bound on the number of rows needed. Efficiency is not a requirement.

(b) Now also rotations of blocks are allowed. Describe how you can adjust your model to this variant.

The following graph shows an optimal solution that fills seven rows for this variant.



3. Miscellaneous Questions

(a) Consider the following constraint:

$$x + 2y \leq 4 \text{ if and only if } x - y \leq 1.$$

Here, x and y are integer variables in the range $\{-10, -9, \ldots, 9, 10\}$.

Rewrite this constraint as integer linear constraints. Note that "$<$" and "$>$" are not allowed. Furthermore, note that you should rewrite the constraint *exactly*, i.e., without any "ε" anywhere.

(b) Consider the following optimization problem:

$$\begin{aligned}
\text{minimize } & \frac{c^\mathsf{T} x}{y+1} \\
\text{subject to } & Ax + ay = b, \\
& 0 \leq x \leq M, \\
& y \in \{0,1\}.
\end{aligned}$$

Note that x is a vector with range $[0,M]^d$, $a \in \mathbb{R}^n$ is a vector, and y is a single binary variable.

Rewrite this optimization problem as a mixed integer program.

(c) Let $G = (V,E)$ be a graph, not necessarily bipartite. We consider the matching polytope

$$\begin{aligned}
\text{minimize } & \sum_{e \in E} w_e x_e \\
\text{subject to } & \sum_{e \in E : v \in e} x_e = 1 \qquad \text{for all } v \in V, 0 \leq x_e \leq 1 \qquad \text{for all } e \in E.
\end{aligned}$$

Assume that we solve the LP with the simplex method and obtain an optimal solution $x^\star$. Prove or disprove: The graph $G^\star = (V, E^\star)$ with $E^\star = \{e \mid 0 < x^\star_e < 1\}$ (this means that $E^\star$ contains all edges that have fractional value in $x^\star$) does not contain cycles of even length.