Exam Queueing Theory
Tuesday, May 31, 2016, 13.30 – 16.30.

---

1. Jobs arrive at a single machine according to a Poisson process with a rate of 8 jobs per hour. The machine processes jobs in order of arrival at an exponential rate of 14 jobs per hour.

   a) Determine the mean sojourn time of a job.

   b) Determine the long-run fraction of jobs with a sojourn time greater than 2 times the mean sojourn time.

   It is possible to increase the production capacity by adding an extra, but old, machine. This machine processes jobs at an exponential rate of 2 jobs per hour. Now assume that both machines are being used, and that, when a job arrives at an empty system, this job will be sent to the fast machine.

   c) The state of the system can be described by n, the number of jobs in the system; when there is only 1 job in the system, the machine processing the job has to be included in the state description, so $(1, f)$ (resp. $(1, s)$) is the state with 1 job on the fast (slow) machine. Give the equilibrium equations and verify by substitution that

   $$p_0 = \frac{7}{27}, \quad p_{1,f} = \frac{3}{27}, \quad p_{1,s} = \frac{7}{27}, \quad p_n = \frac{5}{27}\left(\frac{1}{2}\right)^{n-2}, \quad n \geq 2.$$

   d) Determine the mean sojourn time of a job. Is it a good decision to add the old machine?

2. Customer orders for the production of 1 product arrive at a machine according to a Poisson process with a rate of 1 order per 4 hours. The production of a product consists of two phases: The first phase depends on the specific requirements of the customer. This first phase takes an exponentially distributed time with a mean of 2 hours. The second phase is standard and the same for all customers. This second phase takes a constant time of 1 hour. Orders are processed in order of arrival.

   a) Determine the distribution function and Laplace-Stieltjes transform of the processing time of a product in hours.

   b) Determine the density of the residual processing time in hours and show that the Laplace-Stieltjes transform of the residual processing time in hours is given by

   $$\widetilde{R}(s) = \frac{1 + 2s - e^{-s}}{3s(1 + 2s)}.$$

   c) Show that the Laplace-Stieltjes transform of the waiting time in hours is given by

   $$\widetilde{W}(s) = \frac{s(1 + 2s)}{(4s - 1)(1 + 2s) + e^{-s}}.$$

   d) Determine the mean waiting time in hours.

3. In a waiting room in a hospital patients arrive according to a Poisson process with a rate of λ patients per hour. Nurses arrive at the waiting room according to a Poisson process with rate μ nurses per hour to pick up patients and to bring them to their corresponding doctors. In general, a nurse picks up two patients at a time. However, whenever the nurse arrives at the waiting room and there is only one patient waiting, she just picks up the single patient. If there are no persons waiting when the nurse arrives at the waiting room, she leaves immediately without patients.

   a) What conditions on λ and μ are necessary to guarantee that the system is stable?

   In the sequel suppose $\lambda = 10$ and $\mu = 9$.

   b) Determine the limiting probabilities for the number of patients in the waiting room.

   c) What is the mean time patients spend in the waiting room?

   d) What is the expected number of times per hour that the nurse is picking up two patients, only one patient, and no patients at all, respectively?

4. Jobs arrive at a machine according to a Poisson process with a rate of 10 jobs per hour. Each job requires one or two independent exponential operations, each with a mean of 3 minutes; 40% of the jobs needs just one operation and 60% of the jobs needs two.

   a) Determine the mean production lead time (waiting plus processing) of an arbitrary job when jobs are served in order of arrival.

   b) Determine the mean production lead time, both of jobs requiring only one operation and of those requiring two operations, in the case that jobs requiring just one operation receive non-preemptive priority over the ones that need two operations.

   In the sequel, assume that the machine is immediately turned off when there is no work. It is turned on again when a job arrives; turning on the machine takes an exponential time with a mean of 10 minutes.

   c) Determine the mean production lead time (waiting plus processing) of an arbitrary job when jobs are served in order of arrival.

   d) Determine the mean production lead time, both of jobs requiring only one operation and of those requiring two operations, in the case that jobs requiring just one operation receive non-preemptive priority over the ones that need two operations.

Credits:

1a	b	c	d	2a	b	c	d	3a	b	c	d	4a	b	c	d
2	3	3	2	3	2	3	2	2	3	2	3	2	3	2	3