Exam Queueing Theory — June 19, 2019 (13.45 – 16.45)

This exam consists of five problems. Please put your name and student number on each sheet of paper. Using a simple (not graphic) scientific calculator is allowed. Motivate your answers.

1. Consider the $M/G/1$-FIFO queue with Poisson arrival rate λ and LST (Laplace-Stieltjes Transform) of the service time $\widetilde{B}(s)$.

   a. Let $L(t)$ be the number of jobs in the system at time t, and let L_k^d be the number of jobs left behind in the system just after the k-th departure.
   Argue why $d_n = \lim_{k\to\infty} P(L_k^d = n)$ equals $p_n = \lim_{t\to\infty} P(L(t) = n)$.

   b. Using the probability generating function $P_L(z) = \frac{(1-\rho)\widetilde{B}(\lambda-\lambda z)(1-z)}{\widetilde{B}(\lambda-\lambda z)-z}$ of the number of jobs in system, derive the LST $\tilde{W}(s)$ for the waiting time.

   c. Now assume that $\widetilde{B}(s) = \frac{\mu}{\mu+s}$. Use Mean Value Analysis to determine the expected waiting time of a customer.

2. In a workshop, a machine produces goods upon request. The interarrival times between requests are i.i.d. and are composed of two independent parts. The first part is exponentially distributed with mean 15 minutes. The second part is either exponential with mean 1 hour (with probability 1/2) or it is simply 0 (also with probability 1/2). When the machine is busy, arriving products wait in a queue. The service times at the machine are i.i.d. and exponentially distributed with mean 15 minutes.

   a. Show that the LST of the interarrival times is given by $\tilde{A}(s) = \frac{2s+4}{s^2+5s+4}$.

   b. Determine the probability that a request is served immediately upon arrival.

3. Customers arrive at a 24/7 hairdresser according to a Poisson process with rate 4 per hour, and have an exponential service time distribution with mean 20 minutes. Since there are two assistants working in the shop, two customers can be served simultaneously. When a customer arrives when all assistants are busy, he or she leaves immediately. For each customer that is served, an average profit is made of 7 euros.

   (a) Determine the probability that an arriving customer is not served.

   (b) What is the long-run expected profit per hour?

   At some day the manager decides to offer customers a free cup of coffee in case they arrive when all assistants are busy. As a result, all these customers decide to wait instead of leaving.

   (c) Determine the probability that a customer does not have to wait upon arrival (and hence does not get any coffee).

(d) What is the long-run expected profit per day, if serving coffee costs the manager one euro per cup?

4. Jobs of type c, $c = 1, \ldots, C$, arrive to a service center according to Poisson processes with rate $\lambda(c)$, $c = 1, \ldots, C$. Each job at the service center is served by its own server (for which the service center has an unlimited number of servers available).

   a. Let the service time of a job of type c have an exponential distribution with mean $1/\mu(c)$. Derive the joint equilibrium distribution of the number of jobs of type c, $c = 1, \ldots, C$.

   b. Now assume that the service time of a job of type c is deterministic with length $1/\mu(c)$. Show that the equilibrium distribution of the number of jobs of type c, $c = 1, \ldots, C$, is the same as that in part a.

5. Consider a tandem network of 3 single-server queues. Jobs arrive at queue 1 according to a Poisson process with rate λ. The service times of jobs in queue i are iid and have exponential distributions with rates μ_i, $i = 1, 2, 3$. Jobs that are served in queue 1 route to queue 2, jobs that are served in queue 2 route to queue 3, and jobs served in queue 3 leave the network.

   a. Model this system as a Jackson network of queues and give the state space and the transition rates of the Markov chain that records the number of customers in the queues. Give the stability condition for this network.

   b. Under the stability condition, give the equilibrium distribution of the network. Show that this distribution is correct using Burke's theorem.

   c. Under the stability condition, show that the time it takes a customer to pass through the network has mean $\sum_{j=1}^{3}(\mu_j - \lambda)^{-1}$ and variance $\sum_{j=1}^{3}(\mu_j - \lambda)^{-2}$.

   d. Assume that the total number of customers that is allowed in the network is restricted not to exceed N. A job that arrives to the network and finds N jobs in the network will leave the network and will not return. Give the equilibrium distribution of the network. Show that this distribution is correct using partial balance. Why can't you use Burke's theorem to show this result?

   e. Now assume that only the number of jobs in queue 3 is restricted not to exceed N_3. A job that completes service in queue 2 when queue 3 already contains N_3 jobs will leave the network (and hence will not receive service in queue 3). Give the equilibrium distribution of the network. Show that this distribution is correct.

Norm: Exam grade = total/4

1			2		3				4		5					total
a	b	c	a	b	a	b	c	d	a	b	a	b	c	d	e	
2	3	2	2	3	2	2	2	2	2	2	2	2	3	3	2	+ 4 = 40