1 a) (1) $\rho = \lambda \cdot \mathbb{E}B = \lambda \sum_{k=1}^{\infty} k p_k / \nu = \lambda \mu / \nu$

Note: in exercise $p_k = \frac{\mu^k}{k!} e^{-\mu}$ $k = 1, 2, \ldots$ should be $k = 0, 1, 2, \ldots$

(1) ρ is amount of work entering the system per time-unit
is fraction of time the server is working
fraction of customer that finds the server busy (PASTA)

(2) b) $\tilde{B}(s) = \underbrace{\sum_{k=0}^{\infty} p_k \left(\frac{\nu}{\nu + s}\right)^k}_{(1)} = \underbrace{e^{-\mu\left(1 - \frac{\nu}{\nu + s}\right)} = e^{-\mu \frac{s}{\nu + s}}}_{(1)}$

c) Note that the sojourn time of a customer in LIFO-PR is the time until all customers that entered between its arrival and departure have left plus its own service time. Hence this sojourn time is distributed as the length of a busy period in the M/G/1 queue.

($1\frac{1}{2}$)

From eq (7.20):

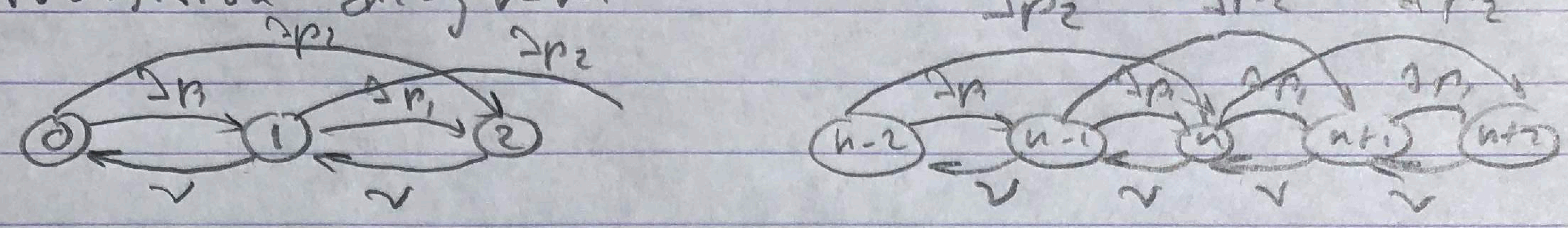
($1\frac{1}{2}$) $\tilde{S}(s) = \tilde{B}(s + \lambda - \lambda \tilde{S}(s))$

d) ($1/2$) $\mathbb{E}S = -S'(s)\big|_{s=0} \Rightarrow \mathbb{E}S = \frac{\mathbb{E}B}{1 - \rho}$

($1/2$) Little's law: $\mathbb{E}L = \lambda \mathbb{E}S = \frac{\lambda \mathbb{E}B}{1 - \lambda \mathbb{E}B}$

(1) depends on $\mathbb{E}B$, only.

(2) Transition diagram



a) $\lambda \underbrace{(p_1 + 2p_2)} < \nu$

(1)

mean amount of parts per customer,

(1) so $\lambda(p_1+2p_2)$ mean amount of parts arriving per time unit

b) Global balance equations, for $n \geq 0$

$$\pi(n)\left(\lambda + \nu\, 1(n>0)\right) = \pi(n-2)\,\lambda p_2\, 1(n>1)$$

(1) $$+\, \pi(n-1)\,\lambda p_1\, 1(n>0)$$

$$+\, \pi(n+1)\,\nu$$

For $n \geq 2$, test solution $\pi(n) = x^n$, gives

$$x^3 \nu - x^2(\lambda + \nu) + x\lambda p_1 + \lambda p_2 = 0$$

$$x = 1, \quad x_{\pm} = \frac{\lambda \pm \sqrt{\lambda^2 + 4\nu\lambda p_2}}{2\nu}$$

both $|x_{\pm}| < 1$ since $\lambda(p_1+2p_2) < \nu$

(1) $\pi(n) = c_+ x_+^n + c_- x_-^n \qquad n \geq 2$

2b) continued   Find c_+, c_- from

(1) $$\sum_{n=0}^{\infty} \pi(n) = 1$$

and boundary equation

2c) $\pi(n)$ is known, see b)

(1) $$\tilde{W}(s) = \sum_{n=0}^{\infty} \pi(n) \left(\frac{\nu}{\nu+s}\right)^n$$

arriving custo[mer] finds n parts in system upon arrival (PASTA)

(1/2)

(1) $$\tilde{B}(s) = p_1 \frac{\nu}{\nu+s} + p_2 \left(\frac{\nu}{\nu+s}\right)^2$$

(1/2) so $\tilde{S}(s) = \tilde{W}(s)\, \tilde{B}(s)$

③ Let $\widetilde{BP}(s)$ LST of BP in M/G/1-FIFO

For this system during the first T time units $N \sim P(\lambda T)$ customers arrive. Each of these customers generates a Busy period

(this can be seen by considering the system under LIFO-PR)

Let $\widetilde{BPT}(s)$ denote the LST of the BP of our system, then

$$\widetilde{BPT}(s) = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} e^{-\lambda T} \left(\widetilde{BP}(s)\right)^{n+1} \cdot \frac{T}{s}$$

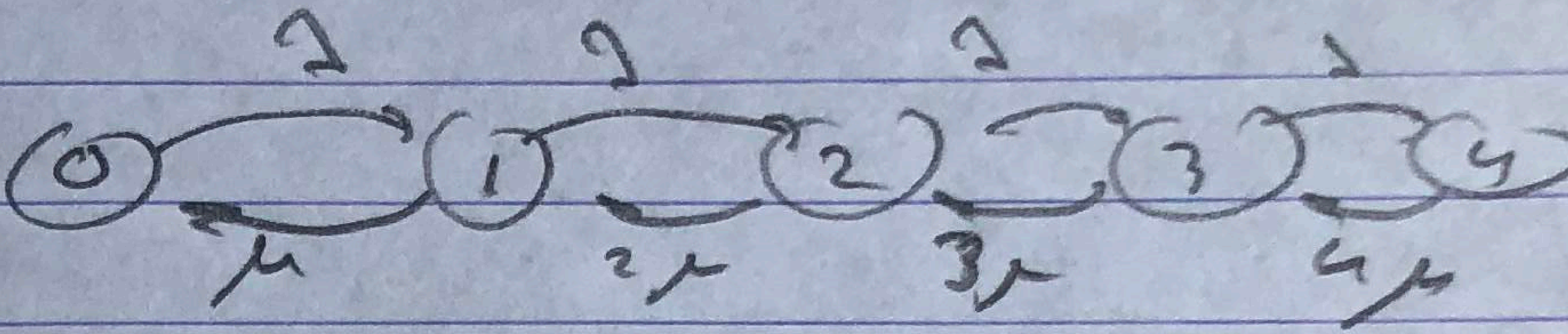
$$= \widetilde{BP}(s) \cdot e^{-\lambda T(1-\widetilde{BP}(s))} \cdot \underbrace{\frac{T}{s}}_{\text{LST of constant } T}$$

Alternatively you may extend the length of the service time B_1 of the first customer by the time T

(4) $M|M|4|4$ system

(1) $\lambda = 4$

$\mu = 2$



(1/2) $$\pi(n) = \frac{(\lambda/\mu)^n}{n!} \Bigg/ \sum_{k=0}^{4} \frac{(\lambda/\mu)^k}{k!}$$

(1) $\pi(4)$ is fraction of time system is full <u>and</u> from PASTA the fraction of customers that is rejected

(1/2) $\pi(4) = \frac{2}{21} < 10\%$ so number of lines is sufficient

(5) Let $x_j = (x_j(1), \ldots, x_j(n_j))$ state of queue j

(1/2) with $x_j(\ell) = t(\ell)$ type of customer in position ℓ

$t(\ell) \in \{1, \ldots, T\}$

Arrival rate λ_t (note t omitted in exercise)

service rates μ_i, $i = 1, 2, 3$ same for all types at the queues

a) Stop-protocol

(1½)

if no types then $-1\frac{1}{2}$

$$
q(x, x') = \begin{cases}
\lambda_t & x_1' = (x_1, t), \quad n_2 < C_2 \\
 & x_2' = x_2 \\
 & x_3' = x_3 \\
\mu_1 & x_1' = (x_1(2), \ldots, x_1(n_1)) \quad n_2 < C_2 \\
 & x_2' = (x_2, x_1(1)) \\
 & x_3' = x_3 \\
\mu_2 & x_1' = x_1 \\
 & x_2' = (x_2(2), \ldots, x_2(n_2)) \\
 & x_3' = (x_3, x_2(1)) \\
\mu_3 & x_1' = x_1 \quad n_2 < C_2 \\
 & x_2' = x_2 \\
 & x_3' = (x_3(2), \ldots, x_3(n_3))
\end{cases}
$$

(1) stability condition $\lambda T < \mu_i$, $i = 1, 2, 3$

5 b) Queue 1 and queue 3 in isolation
1 are quasi-reversible

Queue 2 is <u>not</u> quasi-reversible since
2 the arrival process is <u>stopped</u> if $n_2 = c_2$
so that the arrival process is not
a Poisson process

c) jump-over protocol

$$q(x, x') = \begin{cases} \lambda_t & x_1' = (x_1, t) \\ & x_2' = x_2 \\ & x_3' = x_3 \\ \mu_1 & x_1' = (x_1(2), \dots, x_1(n_1)) \quad n_2 < c_2 \\ & x_2' = (x_2, x_1(1)) \\ & x_3' = x_3 \\ \mu_1 & x_1' = (x_1(2), \dots, x_1(n_1)) \\ & x_2' = (x_2(2), \dots, x_2(n_2), x_1(1)) \\ & x_3' = (x_3, x_2(1)) \\ & \qquad n_2 = c_2 \\ \mu_2 & x_1' = x_1 \\ & x_2' = (x_2(2), \dots, x_2(n_2)) \\ & x_3' = (x_3, x_2(1)) \\ \mu_3 & x_1' = x_1 \\ & x_2' = x_2 \\ & x_3' = (x_3(2), \dots, x_3(n_3)) \end{cases}$$

3

(5) d/ Queue 1 Queue 3 in isolation

(1) quasi-reversible

Queue 2 has Poisson arrival independent of the state of the

(2) queue

Queue 2 in reversed time has the same transition rates, hence is the same Markov chain

Hence, queue 2 has Poisson departures independent of the state of the queue

$\Rightarrow$ Queue 2 is quasi-reversible