Exam Queueing Theory — April 6, 2021 (13.45 – 16.45)

This is an open book exam. This means that you are allowed to use

- reader Queueing Systems
- reader Markovian Queueing Networks
- the slides of the lectures

In addition, you are allowed to use an ordinary (scientific) calculator, but a programmable or graphic calculator ('GR') is not allowed.
You are **not** allowed to use the Internet or any other reference during the exam.
At the end of the time reserved for the exam, or earlier if you decide to stop earlier, you must upload a scan of photos of your results via Canvas as a single pdf file and also send this file via email to r.j.boucherie@utwente.nl. So you have to upload in Canvas and send the email.
Clearly, the pdf file should contain scan or photos should of good quality, so that we can read your results. Scans or photos of insufficient quality will not be marked.
Please read the following paragraph carefully.
By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.
Text to be copied and signed:
I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.

This exam consists of four problems. Please put your name and student number on each sheet of paper. Using a simple (not graphic) scientific calculator is allowed. Motivate your answers.

1. Customers arrive to a shop according to a Poisson process at rate 2 per hour. Arriving customers wait in order of arrival if another customer is already in service. When the service of a customer starts, two tasks are generated that are processed independently and simultaneously. The time it takes to perform one task is exponentially distributed with mean 10 minutes, and the service time of the customer is over when both tasks have been done.

   a. Show that the LST (Laplace-Stieltjes Transform) of the total service time of any customer (counting in hours) is given by

   $$\tilde{B}(s) = \frac{72}{(12+s)(6+s)}.$$

   b. Show that the LST of the sojourn time of any customer is given by

   $$\tilde{S}(s) = \frac{9}{\sqrt{7}} \left(\frac{1}{s+8-2\sqrt{7}} - \frac{1}{s+8+2\sqrt{7}} \right).$$

   c. Find the fraction of customers with sojourn time longer than 30 minutes.

   d. Determine the expected number of customers in the shop.

2. Consider the Erlang loss queue with s servers to which customers of class c arrive according to a Poisson process with rate λ_c, $c = 1, 2, \ldots$. (An arriving customer that finds s servers occupied is blocked and cleared.) The service requirement of customers of class c is Erlang(c, ν) (i.e., consists of c exponential phases with rate ν), $c = 1, 2, \ldots$.

   a. Give a complete description of the Markov chain that records the state (number of customers, classes, remaining service requirements) of the Erlang loss queue (i.e., state space, transition rates.

   b. Give the equilibrium distribution, and prove correctness of the equilibrium distribution.

   c. Give the equilibrium distribution of the total number of customers in the Erlang loss queue.

   d. What is the fraction of class c customers that is blocked and cleared?

3. Consider a queue that serves customers in order of arrival. Customers arrive according to a Poisson process with rate λ and have a generally distributed service requirement with mean $1/\mu$. Assume that $\lambda < \mu$. The server switches off when the system becomes empty, and the server switches on at the moment when the system contains 3 customers (i.e., at the moment the 3rd customer joins the system following an idle period). Give the LST of the consecutive time that the system contains customers.

4. Consider a tandem network of 3 single-server FIFO stations. Stations 1 and 3 have infinite capacity. Station 2 has finite capacity c_2 (so station 2 may contain at most c_2 customers). Customers of type t arrive at station 1 according to a Poisson process with rate λ_t, $t = 1, \dots, T$. The service times of jobs of type t in station i are independent and identically distributed and have an exponential distribution with rate $\mu_{it} = \mu_i$, $i = 1, 2, 3$, $t = 1, \dots, T$. Customers that complete service in station i route to station $i+1$, and customers that complete service in station 3 leave the network. If a customer arrives at station 2 when the number of customers in station 2 equals c_2, then this customer also joins the tail of the queue. In this case, and service of the customer in service at station 2 is interrupted and this customers immediately leaves the server and routes to station 3.

   a. Model this system as a network of stations with multiple customer types. Give the state space and the transition rates of the Markov chain that records the states of the network of stations (including the types of the customers in each position in the queues).

   b. Give the stability condition for this network.

   c. Prove that stations 1, 2 and 3 are quasi-reversible.

   d. Give the equilibrium distribution of the Markov chain that records the types of the customers in each position in the stations and prove correctness.

Norm: Exam grade = total/4

1				2				3	4				total
a	b	c	d	a	b	c	d		a	b	c	d	
2	3	2	3	2	4	2	1	6	2	1	6	2	+ 4 = 40