Exam Queueing Theory – April 12, 2022 (13.45 – 16.45)

This exam consists of five problems. Please put your name and student number on each sheet of paper. Using a simple (not graphic) scientific calculator is allowed. Motivate your answers.

1. Consider the $M/G/1$-FIFO queue with Poisson arrival rate $\lambda = 1$ per hour service time B with LST (Laplace-Stieltjes Transform) $\widetilde{B}(s)$, and mean service time $\mathbb{E}[B]$. Let $\rho = \lambda\mathbb{E}[B]$. Assume that $\rho < 1$.

   a. **[1 pt]** Give two interpretations of ρ.

   b. **[2 pt]** Let $L(t)$ be the number of jobs in the system at time t, and let L_k^d be the number of jobs left behind in the system just after the k-th departure.
   Argue why $d_n = \lim_{k\to\infty} P(L_k^d = n)$ equals $p_n = \lim_{t\to\infty} P(L(t) = n)$.

   c. **[4 pt]** Show that the probability generating function of L, the limiting random variable of $L(t)$, is

$$P_L(z) = \frac{(1-\rho)\widetilde{B}(\lambda - \lambda z)(1-z)}{\widetilde{B}(\lambda - \lambda z) - z}. \tag{1}$$

   d. **[2 pt]** Suppose that the service process is as follows. Each customer receives two operations. The first operation takes an exponential time with a mean of 15 minutes. The second operation is done immediately after the first one and it takes an exponential time with a mean of 20 minutes. Give the LST of the service time.

   e. **[2 pt]** For the service process of d., obtain the mean number of customers in the system in equilibrium.

2. **[3 pt]** Consider the $G/M/1$ queue. The interarrival times A have LST $\widetilde{A}(s)$ with mean $1/\lambda$, and the service times are exponentially distributed with mean $1/\mu$. Assume that $\rho = \lambda/\mu < 1$. Recall that the limiting probability that there are n customers in the system just before an arrival is $a_n = (1-\sigma)\sigma^n$, $n = 0, 1, 2, \ldots$, with σ the unique root in $(0, 1)$ of $\sigma = \widetilde{A}(\mu - \mu\sigma)$.
   Use mean value analysis to obtain $\mathbb{E}[L]$, the mean number of customers in the system at an arbitrary time in equilibrium.

3. At a small river cars are brought from the left side to the right side of the river by a ferry. On average 15 cars per hour arrive according to a Poisson process. It takes the ferry an exponentially distributed time with a mean of 3 minutes to cross the river and return. The capacity of the ferry is equal to 2 cars. The ferry only takes off when there are two or more cars waiting.

   a. **[2 pt]** Give the transition diagram.

   b. **[4 pt]** Determine the distribution of the number of cars that are waiting for the ferry.

4. Jobs of class c, $c = 1, \ldots, C$, arrive to a service center according to Poisson processes with rate $\lambda(c)$, $c = 1, \ldots, C$. The service center has a single server and uses the LIFO-PR queue discipline.

   a. [**2 pt**] Let the service time of a job of class c have an exponential distribution with mean $1/\mu(c)$. Give a description of the Markov chain recording the state of the service center. (Give the states recording the classes of the customers in each position, the state space, and the transition rates.)

   b. [**3 pt**] Give the equilibrium distribution. Use Kelly's lemma to prove that this distribution is indeed the equilibrium distribution.

   c. [**2 pt**] Show that the service center is quasi-reversible.

5. Consider an open network of 3 single-server queues. Jobs arrive at queue 1 according to a Poisson process with rate μ_0. The service times of jobs in queue i are iid and have exponential distributions with rates μ_i, $i = 1, 2, 3$. Jobs that are served in queue 1 route to queue 2 with probability 1/3 and to queue 3 with probability 2/3, jobs served in queue 2 route to queue 3, and jobs served in queue 3 leave the network.

   a. [**2 pt**] Model this system as a Jackson network of queues. (Give the states, the state space and the transition rates of the Markov chain that records the number of customers in the queues.)

   b. [**2 pt**] Give the traffic equations and solve these equations. Give the stability condition for this network.

   c. [**2 pt**] Under the stability condition, give the equilibrium distribution of the network. Can you prove this result using Burke's theorem?

   d. [**3 pt**] Now assume that the number of jobs in queue 3 is restricted not to exceed N_3. A job that arrives at queue 3 when this queue already contains N_3 jobs will leave the network (and hence will not receive service in queue 3). Give the equilibrium distribution of the network. Prove that this distribution is correct.

Norm: Exam grade = total/4

1					2	3		4			5				total
a	b	c	d	e		a	b	a	b	c	a	b	c	d	
1	2	4	2	2	3	2	4	2	3	2	2	2	2	3	+ 4 = 40