Exam Queueing Theory — June 28, 2022 (13.45 – 16.45)

This exam consists of three problems.
Please put your name and student number on each sheet of paper.
Using a simple (not graphic) scientific calculator is allowed.
Motivate your answers.

1. Consider the $M/G/1$-FIFO queue. Let the Poisson arrival process have rate λ. Let the service time distribution be a mixed Erlang distribution that is with probability p_k the sum of k exponentials with rate ν, $k = 1, 2, \ldots$. Let $\sum_{k=1}^{\infty} kp_k \frac{\lambda}{\nu} < 1$.

   a. Give the traffic intensity ρ, and give two different interpretations of this quantity.

   b. Give the LST of the service time distribution.

   c. Give Lindley's equation for the waiting time.

   d. Derive the Pollackzek-Khintchine formula for the LST of the distribution of the waiting time, i.e.,

   $$\widetilde{W}(s) = \frac{(1-\rho)s}{s - \lambda + \lambda \widetilde{B}(s)}. \tag{1}$$

   [Hint: condition on the sojourn time and on the interarrival time.]

   e. Now assume that $p_k = (1-q)q^{k-1}$, $k = 1, 2, \ldots$, with $0 < q < 1$. Give an explicit expression for the LST of the distribution of the waiting time.

   f. Assume that $p_1 = 1$, $p_k = 0$, $k = 2, 3, \ldots$. Use Mean Value Analysis to determine the expected waiting time of a customer.

2. Consider a service center at some company, where customers arrive according to a Poisson process with rate 20 per hour. There are three employees dedicated to serving customers. Customers that arrive while all three employees are busy, will leave the system (and will not return).

   a. Assume that the service times of the customers have an exponential distribution with mean 6 minutes. Determine the equilibrium distribution of the number of customers in the service center. What fraction of arriving customers finds all servers busy in the long run?

   b. Now assume that the service times have an Erlang distribution with 6 phases, each phase with mean 1 minute. Show that the equilibrium distribution of the number of customers in the service center depends on the service times only through their mean.

P.T.O.

The manager decides that rejecting customers is not a good idea, and introduces a waiting room. Customers that arrive when all three employees are busy can now enter the waiting room. Assume that all customers decide to wait when the servers are busy and that customers are served in order of arrival.

8/27 c. Assume that the service times of the customers have an exponential distribution with mean 6 minutes. Determine the equilibrium distribution of the number of customers in the service center. What fraction of arriving customers has to wait in the long run?

2 d. What is the expected waiting time of a customer, given he/she has to wait?

3. Consider a tandem network of 3 queues. Customers of 2 types arrive according to Poisson processes with rate 10 per hour for type 1 and 20 per hour for type 2. Customers in the first queue are served in order of arrival, whereas the second queue uses the LIFO-PR discipline and the third queue the PS discipline. Let the service requirement of customers of type i in queue j be exponential with rates μ_{ij}, $i = 1, 2$, $j = 1, 2, 3$.

   a. Model this system as a network of queues and give the state space and the transition rates of the Markov chain that records the type of the customers in each position in the queues.

   b. Give the stability condition for this network. Which additional condition is required to obtain an explicit expression for the equilibrium distribution of the number of customers of type 1 and 2 in queue 1?

   c. Under the conditions determined in [b.], show that the departure processes of customers of type 1 and 2 from queue 1 are Poisson processes.

   d. Give the transition rates of the time-reversed Markov chain.

   e. Give the equilibrium distribution of the Markov chain that records the type of the customers in each position in the queues and use Kelly's lemma to prove correctness.

Norm:

1						2				3					total
a	b	c	d	e	f	a	b	c	d	a	b	c	d	e	
2	2	2	4	3	2	2	4	2	2	2	2	2	2	3	+ 4 = 40

Exam grade = total/4