Exam Queueing Theory – April 11, 2023 (13.45 – 16.45)

This exam consists of four problems. Please put your name and student number on each sheet of paper. Using a simple (not graphic) scientific calculator is allowed. Using your cheatsheet is allowed. Motivate your answers.

1. Products arrive at a machine to receive service. Each service consists of two consecutive operations: the first takes an exponential amount of time with mean 15 seconds, the second operation takes an exponential amount of time with mean 20 seconds (thus, the server is alternating between type 1 and type 2 operations during a busy period). The arrivals occur as a Poisson process at rate 1 per minute, and products are served in order of arrival.

   a. Determine the LST (Laplace-Stieltjes Transform) of the total service time of a product (counting in minutes).

   b. Show that the LST of the waiting time of any product is given by

   $$\tilde{W}(s) = \frac{5}{12} + \frac{15}{24}\frac{1}{s+1} - \frac{1}{24}\frac{5}{s+5}.$$

   c. Find the distribution function of the waiting time in stationarity.

   d. Determine the expected sojourn time.

2. a. Derive the formula for the LST of the Busy Period in an M/G/1 queue, i.e.,

   $$\tilde{BP}(s) = \tilde{B}(s + \lambda - \lambda \tilde{BP}(s))$$

   Hint: first condition on the service time of the customer that initiates the busy period.

   b. Show that the mean busy period in an M/M/1 queue with arrival rate λ and service rate μ is given by

   $$\mathbb{E}[BP] = \frac{1/\mu}{1-\rho}, \qquad \text{where } \rho = \lambda/\mu.$$

3. Consider an infinite server queue to which customers of class $c = 1, \ldots, C$ arrive according to Poisson processes with rates $\lambda_1, \ldots, \lambda_C$. Customers of class c have mean service requirement $1/\mu_c$, $c = 1, \ldots, C$.

   a. Assume that the service requirement of class c has an exponential distribution, $c = 1, \ldots, C$. Find the equilibrium distribution $\pi(n_1, \ldots, n_C)$ of the number of customers of class $1, \ldots, C$.

   b. Assume that the service requirement of class c has an exponential distribution (i.e., continue with the service requirement assumption of a.). Find the equilibrium distribution $\pi(n)$, where $n = n_1 + \cdots + n_C$ is the total number of customers.

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c. Now assume that the service requirement of customers of class c has an Erlang-c distribution, i.e., is the sum of c exponential phases, $c = 1, \ldots, C$. Find the equilibrium distribution $\pi(r_1, \ldots, r_n)$, where r_i is the remaining number of phases of customer i, $i = 1, \ldots, n$, with $n = n_1 + \cdots + n_C$, and n_c the number of customers of class c, $c = 1, \ldots, C$.

d. Assume that the service requirement of customers of class c has an Erlang-c distribution (i.e., continue with the service requirement assumption of c.). Find the equilibrium distribution $\pi(n)$.

4. Consider an open feedforward network of J quasi-reversible queues in which a customer leaving queue i routes to queue j with probability p_{ij}, $j > i$, or leaves the network with probability p_{i0}. Let $\{N_j(t)\}$ record the state of queue j, with state space S_j, states $\mathbf{n}_j$, and equilibrium distribution π_j, $j = 1, \ldots, J$. Let $\{N(t)\} = \{(N_1(t), \ldots, N_J(t))\}$.

   a. Show that the arrival processes to all queues are Poisson processes.

   b. Show that, in equilibrium, at fixed time t^* the random variable $N_1(t^*), \ldots, N_J(t^*)$ are independent, and conclude from this result that $\pi(\mathbf{n}_1, \ldots, \mathbf{n}_J) = \prod_{j=1}^{J} \pi_j(\mathbf{n}_j)$, $\mathbf{n}_j \in S_j$, $j = 1, \ldots, J$.

   c. Show that $\{N(t)\}$ is quasi-reversible.

   d. Now assume that queue J is modified such that the total number of customers at queue J cannot exceed B. A customer arriving to queue J finding B customers present is discarded and leaves the network. Let S_B denote the state space of queue J taking into account this capacity restriction. Argue that the equilibrium distribution of this network is $\pi(\mathbf{n}_1, \ldots, \mathbf{n}_J) = \prod_{j=1}^{J} \pi_j(\mathbf{n}_j)$, $\mathbf{n}_j \in S_j$, $j = 1, \ldots, J-1$, $\mathbf{n}_J \in S_B$.

<u>Norm:</u> Exam grade =(4+ total)/4

1				2		3				4				total
a	b	c	d	a	b	a	b	c	d	a	b	c	d	
2	3	3	2	3	2	3	2	4	2	2	3	2	3	= 36