Answers Exam Queueing Theory
Monday, May 17, 2010, 13.00–16.00.

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1. a) For machine 1 we have $\lambda_1 = \frac{1}{12}$ (jobs/min) and $\frac{1}{\mu_1} = 4$ (min), so $\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{1}{3}$ and

$$E(S_1) = \frac{\frac{1}{\mu_1}}{1-\rho_1} = 6 \text{ (min)}.$$

Similarly, for machine 2 we have $\lambda_2 = \frac{1}{12}$ (job/min), $\frac{1}{\mu_2} = 8$ (min), so $\rho_2 = \frac{\lambda_2}{\mu_2} = \frac{2}{3}$ and

$$E(S_2) = \frac{\frac{1}{\mu_2}}{1-\rho_2} = 24 \text{ (min)}.$$

For the mean production lead time of an arbitrary job we get

$$E(S) = \frac{1}{2}E(S_1) + \frac{1}{2}E(S_2) = 15 \text{ (min)}.$$

b) For a job on machine 1 it holds

$$P(S_1 > 30) = e^{-\mu_1(1-\rho_1)30} = e^{-5} \approx 0.0067,$$

and for a job on machine 2,

$$P(S_2 > 30) = e^{-\mu_2(1-\rho_2)30} = e^{-5/4} \approx 0.287.$$

Hence, for an arbitrary job,

$$P(S > 30) = \frac{1}{2}P(S_1 > 30) + \frac{1}{2}P(S_2 > 30) = \frac{1}{2}e^{-5} + \frac{1}{2}e^{-5/4} \approx 0.147.$$

c) The mean production lead time of an arbitrary job is equal to

$$\begin{aligned} E(S(p)) &= \frac{p\frac{1}{\mu_1}}{1-p\frac{\lambda}{\mu_1}} + \frac{(1-p)\frac{1}{\mu_2}}{1-(1-p)\frac{\lambda}{\mu_2}} \\ &= \frac{12p}{3-2p} + \frac{24-24p}{4p-1} \\ &= \frac{18}{3-2p} + \frac{18}{4p-1} - 12. \end{aligned}$$

The value of p minimizing the mean production lead time satisfies

$$\frac{d}{dp}E(S(p)) = \frac{36}{(3-2p)^2} - \frac{72}{(4p-1)^2} = 0,$$

so

$$\frac{1}{3-2p} = \frac{\sqrt{2}}{4p-1},$$

and hence, the optimal p is given by

$$p = \frac{1}{4} \cdot \frac{6+\sqrt{2}}{1+\sqrt{2}} \approx 0.768.$$

2. a) The problem can be modeled as an $M/G/s/s$ system, where a busy server represents an outstanding order. The number of books on stock is then s minus the number of busy servers. Here we have $s = 2$, $\lambda = 1$ (customer/day) and $E(B) = 2$ days, so $\rho = \lambda E(B) = 2$. The long-run fraction of time p_i that there are i outstanding orders is

$$p_i = \frac{\frac{\rho^i}{i!}}{\sum_{k=0}^{2} \frac{\rho^k}{k!}} = \frac{\frac{2^i}{i!}}{1+2+2} = \frac{1}{5}\frac{2^i}{i!}, \quad i = 0, 1, 2.$$

So $p_0 = \frac{1}{5}$, $p_1 = p_2 = \frac{2}{5}$. The probability that there is no book on the shelf is $p_2 = \frac{2}{5}$, that there is 1 book is $p_1 = \frac{2}{5}$ and that there are 2 books is $p_0 = \frac{1}{5}$.

b) By PASTA, the long-run fraction of customer demand that is lost is equal to $p_2 = \frac{2}{5}$.

c) When $s = 3$, the long-run fraction of customer demand that is lost is equal to

$$\frac{\frac{2^3}{3!}}{\sum_{k=0}^{3} \frac{\rho^k}{k!}} = \frac{\frac{4}{3}}{\frac{19}{3}} = \frac{4}{19} > \frac{1}{10}.$$

When $s = 4$ this fraction is equal to

$$\frac{\frac{2^4}{4!}}{\sum_{k=0}^{4} \frac{\rho^k}{k!}} = \frac{\frac{2}{3}}{7} = \frac{2}{21} < \frac{1}{10}.$$

Hence, the book store should keep at least 4 copies on stock.

3. As time unit we choose 1 second:

$$\lambda_N = 1/20,\ E(B_N) = 8,\ \rho_N = 2/5,\ E(R_N) = 4,$$

$$\lambda_W = 1/30,\ E(B_W) = 12,\ \rho_W = 2/5,\ E(R_W) = 6.$$

$$\lambda = 1/12,\ E(B) = 3/5 \cdot 8 + 2/5 \cdot 12 = 48/5,\ \rho = 4/5,\ E(R) = 1/2 \cdot 4 + 1/2 \cdot 6 = 5,$$

a) $E(W) = \frac{\rho}{1-\rho} E(R) = 20$ seconds. Hence, $E(S_N) = 28$ and $E(S_W) = 32$ seconds.

b)
$$E(W_N) = \frac{\rho_N E(R_N) + \rho_W E(R_W)}{1 - \rho_N} = 20/3 = 6\frac{2}{3} \text{ seconds },$$

$$E(W_W) = \frac{\rho_N E(R_N) + \rho_W E(R_W)}{(1 - \rho_N)(1 - \rho_N - \rho_W)} = 100/3 = 33\frac{1}{3} \text{ seconds },$$

Hence, $E(S_N) = 44/3 = 14\frac{2}{3}$ seconds and $E(S_W) = 136/3 = 45\frac{1}{3}$ seconds .

c)
$$E(W_N) = \frac{\rho_N E(R_N) + (1 - \rho_N) E(R_W)}{1 - \rho_N} = 26/3 = 8\frac{2}{3} \text{ seconds },$$

Hence, $E(S_N) = 50/3 = 16\frac{2}{3}$ seconds .

d) $(1 - \rho_N) \cdot 5 = 3$ pallets per minute.

4. a)
$$\widetilde{B}(s) = E(e^{-sB}) = e^{-2s} \cdot \frac{1}{1+s}$$

b) We have $E(B) = 3$ hours. Hence,

$$\widetilde{R}(s) = \frac{1 - \widetilde{B}(s)}{sE(B)} = \frac{1 + s - e^{-2s}}{3s(1+s)}.$$

c) We have $\lambda = 1/4$ and $E(B) = 3$ hours, so $\rho = \lambda E(B) = 3/4$. Hence

$$\widetilde{W}(s) = \frac{1-\rho}{1-\rho\widetilde{R}(s)} = \frac{1}{4-3\widetilde{R}(s)} = \frac{s(1+s)}{(4s-1)(1+s)+e^{-2s}}.$$

d) Use

$$E(W) = \frac{\rho}{1-\rho}E(R) = 3E(R)$$

and

$$E(R) = \frac{E(B^2)}{2E(B)} = \frac{5}{3}$$

to obtain $E(W) = 5$ and $E(S) = E(W) + E(B) = 8$ hours.

e) Let

$$\begin{aligned}
p_i &= P(\text{ customer arrives when machine is idle }) = 1/4, \\
p_f &= P(\text{ customer arrives when machine works on first phase }) = 1/2, \\
p_s &= P(\text{ customer arrives when machine works on second phase }) = 1/4.
\end{aligned}$$

Then the mean value equations are given by

$$\begin{aligned}
E(L) &= \lambda E(S) = E(S)/4, \\
E(S) &= (E(L)+1)\cdot 3 - p_i \cdot 2 - p_f \cdot 1 - p_s \cdot 2.
\end{aligned}$$

This gives $E(S) = 6$ hours. Hence, the reduction is 2 hours.
