Answers Exam Queueing Theory
Monday, June 21, 2010, 13.00–16.00.

---

1. a) We are dealing with an $M/M/c$ queue with $c = 4$. As time unit we choose 1 minute: $\lambda = 2/3$, $\mu = 1/3$ and $\rho = 1/2$. As solution of the balance equations and the normalization equation we obtain

$$p_0 = \frac{3}{23}, \quad p_1 = \frac{6}{23}, \quad p_2 = \frac{6}{23}, \quad p_{3+n} = \frac{4}{23}\left(\frac{1}{2}\right)^n, \quad n \geq 0.$$

   b) $E(L) = \sum_{n=0}^{\infty} np_n = 50/23$ and hence $E(S) = E(L)/\lambda = 150/46 \approx 3.26$ minutes.

   c) $P(W > 2) = \Pi_W \cdot e^{-c\mu(1-\rho)2} = \frac{4}{23} \cdot e^{-4/3} \approx 0.046$.

   d) E(crowded period) = E(busy period in $M/M/1$ queue with $\lambda = 2/3$ and $\mu = 4/3$) = $1/[\mu(1-\rho)] = 1.5$ minutes. The fraction of time that the system is crowded equals E(crowded period)/[E(crowded period)+ E(quiet period)]=4/23. So E(quiet period) = $7\frac{1}{8} = 7.125$ minutes.

2. a) Customers arrive with rate $\lambda = \frac{1}{20}$ (customers/hour) and the service time of a job is exponentially distributed with a mean $\frac{1}{\mu} = 6$ (minutes). Two third of the customers have only one job, the remaining one third have two jobs. Hence, $\rho_1 = \frac{2}{3}\lambda\frac{1}{\mu} = \frac{1}{5}$ is the fraction of time the handyman works on customers with one job, and $\rho_2 = \frac{1}{3}\lambda\frac{2}{\mu} = \frac{1}{5}$ is the fraction of time the handyman works on customers with two jobs. The mean waiting time of an arbitrary customer is

$$E(W) = \frac{\rho_1 \cdot \frac{1}{\mu} + \rho_2\left(\frac{1}{2} \cdot \frac{2}{\mu} + \frac{1}{2} \cdot \frac{1}{\mu}\right)}{1 - \rho_1 - \rho_2} = 5 \text{ (minutes)}$$

   and the mean sojourn time of an arbitrary customer is

$$E(S) = E(W) + \frac{2}{3} \cdot \frac{1}{\mu} + \frac{1}{3} \cdot \frac{2}{\mu} = 13 \text{ (minutes)}.$$

   b) The mean number of waiting customers is

$$E(L^q) = \lambda E(W) = \frac{1}{4},$$

   and so the mean number of unfinished jobs in the shop of the handyman is

$$E(L^q) \cdot (\frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 2) + \rho_1 \cdot 1 + \rho_2 \cdot \frac{3}{2} = \frac{5}{6}.$$

   c) The mean waiting time of a customer with one job equals

$$E(W_1) = \frac{\rho_1 \cdot \frac{1}{\mu} + \rho_2\left(\frac{1}{2} \cdot \frac{2}{\mu} + \frac{1}{2} \cdot \frac{1}{\mu}\right)}{1 - \rho_1} = \frac{15}{4} \text{ (minutes)}$$

   and of a customer with two jobs

$$E(W_2) = \frac{\rho_1 \cdot \frac{1}{\mu} + \rho_2\left(\frac{1}{2} \cdot \frac{2}{\mu} + \frac{1}{2} \cdot \frac{1}{\mu}\right)}{(1 - \rho_1)(1 - \rho_1 - \rho_2)} = \frac{25}{4} \text{ (minutes)}$$

d) The mean number of waiting customers with one job is

$$E(L_1^q) = \frac{2}{3}\lambda E(W_1) = \frac{1}{8}$$

and the mean number of waiting customers with two jobs is

$$E(L_2^q) = \frac{1}{3}\lambda E(W_2) = \frac{5}{48}.$$

So the mean number of unfinished jobs in the shop of the handyman is

$$E(L_1^q)\cdot 1 + E(L_2^q)\cdot 2 + \rho_1 \cdot 1 + \rho_2 \cdot \frac{3}{2} = \frac{5}{6}.$$

Of course, this is the same answer as in **b)**, because the number of unfinished jobs does not depend on the order in which the handyman handles the jobs.

3. a) This problem can be described by an $M/G/\infty$ system, where a busy server corresponds to an outstanding order at the supplier. Denote by i the number of busy servers. Then $\max\{0, 3-i\}$ is the number of books on the shelf. Note that $\lambda = 1$ (customer/day) and $E(B) = 2$ days, so $\rho = \lambda E(B) = 2$. The long-run fraction of time p_i of i outstanding orders is

$$p_i = e^{-\rho}\frac{\rho^i}{i!} = e^{-2}\frac{2^i}{i!}, \quad i = 0, 1, 2, \ldots.$$

b) The mean number of books on the shelf is

$$p_0 \cdot 3 + p_1 \cdot 2 + p_2 \cdot 1 = 9e^{-2} \approx 1.22$$

c) The mean number of waiting customers equals

$$\begin{aligned}
\sum_{i=3}^{\infty}(i-3)p_i &= \sum_{i=0}^{\infty}(i-3)p_i + \sum_{i=0}^{2} p_i(3-i) \\
&= \rho - 3 + \sum_{i=0}^{2} p_i(3-i) \\
&= 2 - 3 + 9e^{-2} \\
&\approx 0.22
\end{aligned}$$

and hence, using Little's formula, we obtain $E(W) = 0.22$ days.

4. As time unit we choose 1 minute. The second stage of the production line behaves as a $G/M/1$ queue with $\mu = 1$ and

$$\widetilde{A}(s) = \frac{2}{3}\cdot\frac{1}{1+s} + \frac{1}{3}\cdot\frac{1}{1+3s}.$$

a) The mean interarrival time at the second stage of the production line equals the mean production time at machine 1, and hence it is equal to 5/3. The mean service time at the second stage of the production line equals the mean production time at machine 2, and hence it is equal to 1. We conclude that the fraction of time machine 2 is working equals 3/5.

b) Solving

$$\sigma = \widetilde{A}(\mu(1-\sigma))$$

we obtain that $\sigma = \frac{2}{3}$ is the unique solution in the interval $(0, 1)$. Hence, the sojourn time (in hours) is exponentially distributed with parameter $\mu(1-\sigma) = \frac{1}{3}$.

c) $E(L^a) = \frac{\sigma}{1-\sigma} = 2$.

d) $E(S) = \frac{1}{\mu(1-\sigma)} = 3$ and hence $E(L) = \lambda \cdot E(S) = \frac{3}{5} \cdot 3 = 1\frac{4}{5}$.

5. As time unit we choose 1 minute:

$$\lambda = 1/10, E(B) = 15/2, \rho = 3/4 \text{ and } E(R_B) = 35/9.$$

a) The mean waiting time of a customer equals

$$E(W) = E(L^q) \cdot 15/2 + 1/10 \cdot 15 + 1/10 \cdot 5 + 1/20 \cdot 5/2 + 3/4 \cdot 35/9.$$

Together, with Little's formula $E(L^q) = 1/10 \cdot E(W)$, this gives $E(W) = 121/6 = 20.17$ minutes. Hence, the mean sojourn time of a customer equals $E(S) = 27.67$ minutes.

b) The fraction of time that the server serves customers is 3/4. The mean duration of a period that the server is away equals $10 + 10 + 5 = 25$ minutes. Hence, the mean duration of a busy period equals 75 minutes.
