Answers Exam Queueing Theory
Monday, May 21, 2012, 16.00–19.00.

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1. a) Since $\rho = \frac{4}{7}$,

$$E(S) = \frac{\frac{1}{14}}{1 - \frac{4}{7}} = \frac{1}{6} \text{ (hour)} = 10 \text{ (min)}.$$

b) $$P(S > \frac{1}{3}) = e^{-2} \approx 0.135.$$

c) The balance equations are

$$\begin{aligned} 8p_0 &= 14p_{1,f} + 2p_{1,s} \\ 10p_{1,s} &= 14p_2 \\ 8(p_{1,f} + p_{1,s}) &= 16p_2 \\ 16p_n &= 8p_{n-1}, \quad n > 2. \end{aligned}$$

The solution of this set of equations, together with the normalization equation, is

$$p_0 = \frac{7}{27}, \quad p_{1,f} = \frac{3}{27}, \quad p_{1,s} = \frac{7}{27}, \quad p_n = \frac{5}{27}\left(\frac{1}{2}\right)^{n-2}, \quad n \geq 2.$$

d) $$E(L) = p_{1,s} + p_{1,f} + \sum_{n=2}^{\infty} np_n = \frac{40}{27}, \quad E(S) = \frac{1}{8} \cdot \frac{40}{27} = \frac{5}{27} \text{ (hour)},$$

which is greater than $\frac{1}{6}$. So it's not a good idea to use the old machine!

2. a) Let B denote the processing time plus a possible cleaning time, and let R denote the residual of B. Then it follows that

$$\widetilde{B}(s) = \frac{2(1 + \frac{7}{8}s)}{(2+s)(1+s)}, \quad \widetilde{R}(s) = \frac{2 + \frac{8}{5}s}{(2+s)(1+s)}.$$

Hence, with $\rho = 1 \cdot (\frac{1}{2} + \frac{1}{8}) = \frac{5}{8}$,

$$\begin{aligned} \widetilde{W}(s) &= \frac{1-\rho}{1 - \rho\widetilde{R}(s)} \\ &= \frac{3(2+s)(1+s)}{8s^2 + 16s + 6} \\ &= \frac{3(2+s)(1+s)}{8(s+\frac{1}{2})(s+\frac{3}{2})} \\ &= \frac{3}{8} + \frac{9}{16} \cdot \frac{1}{1+2s} + \frac{1}{16} \cdot \frac{1}{1+\frac{2}{3}s}. \end{aligned}$$

b) Hence,

$$P(W > t) = \frac{9}{16}e^{-\frac{1}{2}t} + \frac{1}{16}e^{-\frac{3}{2}t},$$

so

$$P(W > 2) = \frac{9}{16}e^{-1} + \frac{1}{16}e^{-3} \approx 0.215.$$

c) From a),

$$E(W) = \frac{9}{16} \cdot 2 + \frac{1}{16} \cdot \frac{2}{3} = \frac{7}{6} \text{ (hour).}$$

Or,

$$E(W) = E(L^q)\left(\frac{1}{2} + \frac{1}{8}\right) + 1 \cdot \frac{1}{2} \cdot \left[\frac{1}{2} + \frac{1}{8} \cdot 1\right] + 1 \cdot \frac{1}{8} \cdot 1 = E(L^q) \cdot \frac{5}{8} + \frac{7}{16},$$

so with $E(L^q) = 1 \cdot E(W)$,

$$E(W) = \frac{8}{3} \cdot \frac{7}{16} = \frac{7}{6} \text{ (hour).}$$

Hence

$$E(S) = E(W) + \frac{1}{2} = \frac{5}{3} \text{ (hour).}$$

3. a) The mean interarrival time of a batch is $\frac{2}{3}$ hour; the mean processing time of a batch is $\frac{1}{\mu} = \frac{1}{4}$ hour. Hence, the utilization of the furnace is $\rho = \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}$.

b) The batch process of the furnace can be modeled as an $E_2/M/1$ system. The arrival distribution is

$$a_n = (1-\sigma)\sigma^n, \quad n = 0, 1, 2, \ldots,$$

where σ is the unique root on $(0,1)$ of

$$\sigma = \widetilde{A}(\mu(1-\sigma)) = \left(\frac{3}{3 + 4(1-\sigma)}\right)^2.$$

This yields $\sigma = \frac{1}{4}$.

c) The distribution of the production lead time S of a batch is exponential with mean $\frac{1}{4(1-\sigma)} = \frac{1}{3}$ hour. Hence,

$$P(S > 2 \cdot \frac{1}{3}) = e^{-2} \approx 0.135.$$

d) The production lead time S_p of a pod is the waiting time till the batch is complete plus the sojourn time of a batch. Hence,

$$E(S_p) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot 0 + E(S) = \frac{1}{2} \text{ (hour).}$$

4. a) Denote the short packets by type 1, the long ones by type 2. Then

$$\lambda_1 = \frac{1}{8} \cdot \frac{2}{5} = \frac{1}{20}, \quad B_1 = 1, \quad \rho_1 = \frac{1}{20}, \quad \lambda_2 = \frac{1}{8} \cdot \frac{3}{5} = \frac{3}{40}, \quad B_2 = 10, \quad \rho_2 = \frac{3}{4}.$$

Hence,

$$E(W) = \frac{\rho_1 \cdot \frac{1}{2} + \rho_2 \cdot 5}{1 - \rho_1 - \rho_2} = \frac{151}{8} = 18\frac{7}{8}.$$

b)
$$E(W_1) = \frac{\rho_1 \cdot \frac{1}{2} + \rho_2 \cdot 5}{1 - \rho_1} = \frac{151}{38} = 3\frac{37}{38}, \quad E(W_2) = \frac{\rho_1 \cdot \frac{1}{2} + \rho_2 \cdot 5}{(1-\rho_1)(1-\rho_1-\rho_2)} = \frac{755}{38} = 19\frac{33}{38}.$$

c) The expected time, that the interface is asleep, is equal to $e^{-\frac{1}{8}} \cdot 8$. Hence, for type 1 it follows that, with $\rho = \rho_1 + \rho_2 = \frac{4}{5}$,

$$\begin{aligned} E(W_1) &= \rho_1 \cdot \frac{1}{2} + \rho_2 \cdot 5 + E(L_1^q) \cdot 1 + \\ &\quad +(1-\rho) \cdot \left[\frac{1}{2+8e^{-\frac{1}{8}}} \cdot \left(\frac{1}{2} + 1 \right) + \frac{8e^{-\frac{1}{8}}}{2+8e^{-\frac{1}{8}}} \cdot 1 + \frac{1}{2+8e^{-\frac{1}{8}}} \cdot \frac{1}{2} \right] \\ &= \frac{151}{40} + E(L_1^q) + \frac{1}{5}. \end{aligned}$$

and thus, with Little's law $E(L_1^q) = \lambda_1 E(W_1) = \frac{1}{20} E(W_1)$,

$$E(W_1) = \frac{151}{38} + \frac{20}{19} \cdot \frac{1}{5} = \frac{159}{38} = 4\frac{7}{38}, \quad E(L_1^q) = \frac{159}{760}.$$

For $E(W_2)$ it follows that

$$E(W_2) = \rho_1 \cdot \frac{1}{2} + \rho_2 \cdot 5 + E(L_1^q) \cdot 1 + E(L_2^q) \cdot 10 + \frac{1}{5} + \rho_1 E(W_2),$$

so, with $E(L_2^q) = \lambda_2 E(W_2)$,

$$E(W_2) = 5 \cdot \left(\frac{151}{40} + \frac{159}{760} + \frac{1}{5} \right) = \frac{151}{8} + \frac{159}{152} + 1 \approx 20.92.$$

d) The percentage of energy saving is

$$(1-\rho) \cdot \frac{8e^{-\frac{1}{8}}}{2+8e^{-\frac{1}{8}}} \cdot 0.9 \cdot 100\% \approx 14\%$$
