Answers Exam Queueing Theory
Monday, June 25, 2012, 14.00–17.00.

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1. a) Let p_n be the equilibrium probability of having n guests at the reception. The cut equations are given by

$$\begin{aligned} 12\,p_1 &= 3\,p_0 \\ 12\,p_{n+1} &= 3\,p_n + \tfrac{3}{2}\,p_{n-1}, \quad n = 1, 2 \ldots. \end{aligned}$$

The solution of this set of equations, together with the normalization equation, is

$$p_n = \tfrac{5}{12}\left(\tfrac{1}{2}\right)^n + \tfrac{5}{24}\left(-\tfrac{1}{4}\right)^n.$$

b) We have

$$E(L) = \sum_{n=0}^{\infty} n\,p_n = \tfrac{4}{5}.$$

The average number of guests arriving at the reception is $\lambda = \frac{9}{2}$ guests per hour. Hence, using Little's law we obtain $E(S) = E(L)/\lambda = 8/45$ hours $= 32/3$ minutes and $E(W) = E(S) - E(B) = 17/3$ minutes.

c) The mean waiting time of a customer who arrived in a group of two and is served after the other guest arriving at the same time is given by

$$E(W^{(\text{second})}) = E(L) \cdot 5 + 5 = 9 \text{ minutes}.$$

2. a) With time unit minute we have

$$\lambda_A = \tfrac{3}{60}, \quad E(B_A) = E(R_A) = 8, \quad \rho_A = \tfrac{2}{5}, \quad \lambda_B = \tfrac{5}{60}, \quad E(B_B) = E(R_B) = 4, \quad \rho_B = \frac{1}{3}.$$

The mean value relations are given by

$$\begin{aligned} E(S_A) &= E(L_A)\,8 + E(L_B)\,4 + 8, \\ E(S_B) &= E(L_A)\,8 + E(L_B)\,4 + 4, \\ E(L_A) &= \tfrac{3}{60}E(S_A), \\ E(L_B) &= \tfrac{5}{60}E(S_B). \end{aligned}$$

and have solution

$$E(S_A) = 25,\ E(S_B) = 21,\ E(L_A) = \tfrac{5}{4},\ E(L_B) = \tfrac{7}{4}.$$

Furthermore, we have $E(W_A) = E(W_B) = 17$.

b) We have

$$E(\text{ Work }) = \tfrac{5}{4} \cdot 8 + \tfrac{7}{4} \cdot 4 = 17 \text{ minutes}.$$

c) We have

$$E(S_B) = \frac{\rho_A E(R_A) + \rho_B E(R_B)}{1 - \rho_B} + E(B_B) = 10.8 \text{ minutes },$$

and

$$E(S_A) = \frac{\rho_A E(R_A) + \rho_B E(R_B)}{(1-\rho_B)(1-\rho_A-\rho_B)} + E(B_A) = 33.5 \text{ minutes} .$$

This gives

$$E(W_A) = 25.5,\ E(W_B) = 6.8,\ E(L_A) = \tfrac{67}{40},\ E(L_B) = \tfrac{9}{10}.$$

d) We have

$$E(\text{ Work }) = \tfrac{67}{40}\cdot 8 + \tfrac{9}{10}\cdot 4 = 17 \text{ minutes.}$$

This is the same as in part b. The amount of work in the system is independent of the service order!

3. As time unit we choose 1 hour. The second stage of the production line behaves as a $G/M/1$ queue with $\mu = 2$ and

$$\widetilde{A}(s) = \frac{2}{2+s}\cdot\frac{3}{3+s}.$$

a) The mean interarrival time at the second stage of the production line equals the mean production time at machine 1, and hence it is equal to 50 minutes. The mean service time at the second stage of the production line equals the mean production time at machine 2, and hence it is equal to 30 minutes. We conclude that the fraction of time machine 2 is working equals $\rho = \frac{3}{5}$.

b) Solving

$$\sigma = \widetilde{A}(\mu(1-\sigma))$$

we obtain that $\sigma = \frac{1}{2}$ is the unique solution in the interval $(0,1)$. Hence, the sojourn time (in hours) is exponentially distributed with parameter $\mu(1-\sigma) = 1$.

c) The total mean number of products in the second stage of the production line at an arrival instant is given by $E(L^a) = \frac{\sigma}{1-\sigma} = 1$. The probability that machine 2 is occupied at an arrival instant is given by $\sigma = \frac{1}{2}$. Hence at an arrival instant the mean number of products on machine 2 is $\frac{1}{2}$ and the mean number of products in the buffer between machine 1 and machine 2 is $\frac{1}{2}$.

d) The total mean number of products in the second stage of the production line at an arbitrary instant is given by $E(L) = \frac{\rho}{1-\sigma} = \frac{6}{5}$. The probability that machine 2 is occupied at an arbitrary instant is given by $\rho = \frac{3}{5}$. Hence at an arbitrary instant the mean number of products on machine 2 is $\frac{3}{5}$ and the mean number of products in the buffer between machine 1 and machine 2 is $\frac{3}{5}$.

4. a) This is an $M/G/1$ queue with $\rho = \frac{3}{5}$ and $E(R) = \frac{31}{3}$ minutes. Hence

$$E(S) = \frac{\rho}{1-\rho} \cdot E(R) + E(B) = \frac{55}{2} \text{ minutes .}$$

b) The average length of an idle period is 20 minutes, the average length of a busy period is 30 minutes. Hence the average switch-on costs are 20 euro per 50 minutes, i.e. 24 euros per hour.

c) The arrival relation in the new situation is given by

$$E(S) = [E(L) + 1] \cdot 12 - \rho_2 \cdot 10 - p_{idle} \cdot 10$$

where $p_{idle} = \frac{2}{5}$ is the probability that the machine is switched off and $\rho_2 = \frac{1}{10}$ is the probabililty that the machine is working on a second phase of a product. Together with Little's law $E(L) = \lambda E(S)$ this will lead to $E(S) = \frac{35}{2}$ minutes.
