Answers Exam Queueing Theory
Monday, May 19, 2014, 13.30–16.30.

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1. a) Random splitting of a Poisson process gives two independent Poisson processes. As a consequence, the number of jobs at the two machines are independent of each other.

   b) Both machines behave as $M/M/1$ queues. For machine 1 we have $\lambda_1 = \frac{3}{2}$, $\mu_1 = 2$ and hence $\rho_1 = \frac{3}{4}$. For machine 2 we have $\lambda_2 = \frac{1}{2}$, $\mu_2 = 1$ and hence $\rho_2 = \frac{1}{2}$. Because of the independence of the two systems (see a), we have

   $$p_{i,j} = (1-\rho_1)\rho_1^i \cdot (1-\rho_2)\rho_2^j = \tfrac{1}{8}\left(\tfrac{3}{4}\right)^i \left(\tfrac{1}{2}\right)^j .$$

   c) We have

   $$E(W_1) = \tfrac{\rho_1}{1-\rho_1} \cdot \tfrac{1}{\mu_1} = \tfrac{3}{2} \text{ hour} \quad \text{and} \quad E(W_2) = \tfrac{\rho_2}{1-\rho_2} \cdot \tfrac{1}{\mu_2} = 1 \text{ hour}$$

   and hence

   $$E(W) = \tfrac{3}{4}E(W_1) + \tfrac{1}{4}E(W_2) = \tfrac{11}{8} \text{ hour.}$$

   Furthermore,

   $$P(W > 1) = \tfrac{3}{4}P(W_1 > 1) + \tfrac{1}{4}P(W_2 > 1) = \tfrac{9}{16}e^{-1/2} + \tfrac{1}{8}e^{-1/2} = \tfrac{11}{16}e^{-1/2}.$$

   d) The mean number of crowded periods per day equals $48(p_{0,1}+p_{1,0}) = 48 \cdot \frac{5}{32} = \frac{15}{2} = 7\frac{1}{2}$.

   e) The fraction of time the system is crowded is $1 - p_{0,1} - p_{0,1} - p_{1,0} = \frac{23}{32}$. Hence, the average number of crowded hours per day is equal to $24 \cdot \frac{23}{32} = \frac{69}{4} = 17\frac{1}{4}$. Hence, the expected duration of a crowded period is $\frac{69}{4}/\frac{15}{2} = \frac{23}{10} = 2.3$.

2. a) The machine works with speed 2. On this machine the processing time of a type 1 job is exponential with a mean of $\frac{1}{4}$ hours and the processing time of a type 2 job is exponential with a mean of $\frac{1}{2}$ hours. Hence,

   $$\widetilde{B}(s) = \frac{3}{4} \cdot \frac{4}{4+s} + \frac{1}{4} \cdot \frac{2}{2+s} = \frac{8 + \frac{7}{2}s}{(4+s)(2+s)}.$$

   b) With $\rho = 2 \cdot (\frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2}) = \frac{5}{8}$, we have

   $$\begin{aligned}
   \widetilde{W}(s) &= \frac{(1-\rho)s}{\lambda\widetilde{B}(s) + s - \lambda} \\
   &= \frac{3(4+s)(2+s)}{8(s^2+4s+3)} \\
   &= \frac{3(4+s)(2+s)}{8(s+1)(s+3)} \\
   &= \frac{3}{8} + \frac{9}{16} \cdot \frac{1}{1+s} + \frac{1}{16} \cdot \frac{3}{3+s}.
   \end{aligned}$$

   c) We have

   $$P(W > 1) = \frac{9}{16}e^{-1} + \frac{1}{16}e^{-3} \approx 0.215.$$

d) We have

$$E(W) = \frac{9}{16} \cdot 1 + \frac{1}{16} \cdot \frac{1}{3} = \frac{7}{12} \text{ hour}$$

and from Little's law

$$E(L^q) = \lambda E(W) = \frac{7}{6}.$$

3. The system can be modeled as a $G/M/1$ system. Time unit: minute. The Laplace-Stieltjes transform of the inter-arrival times is given by $\widetilde{A}(s) = \frac{1}{1+4s} \cdot \frac{1}{1+6s}$, the service rate $\mu = \frac{1}{6}$. The mean interarrival time is 10 minutes. The occupation rate of the machine is $\rho = \frac{3}{5}$.

a) The arrival distribution is

$$a_n = (1-\sigma)\sigma^n, \quad n = 0, 1, 2, \ldots,$$

where σ is the unique root on $(0,1)$ of

$$\sigma = \widetilde{A}(\mu(1-\sigma)) = \frac{3}{(5-2\sigma)(2-\sigma)}.$$

This yields $\sigma = \frac{1}{2}$.

b) For the Laplace-Stieltjes transform of the waiting time we have

$$\widetilde{W}(s) = \sum_{n=0}^{\infty} a_n \left(\frac{\mu}{\mu+s}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \left(\frac{1}{1+6s}\right)^n = \frac{1+6s}{1+12s} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{1+12s}.$$

c) We have $P(W > t) = \sigma e^{-\mu(1-\sigma)t}$. Hence $P(W > 12) = \frac{1}{2}e^{-1}$.

4. The arrival rate is $\lambda = \frac{1}{4}$ per hour and the service time $B = U[0, \frac{20}{3}]$ hour. Note that if $X = U[a,b]$, then

$$E(X) = \frac{a+b}{2}, \quad \text{var}(X) = \frac{1}{12}(b-a)^2, \quad c_X^2 = \frac{\text{var}(X)}{(E(X))^2} = \frac{1}{3}\left(\frac{b-a}{b+a}\right)^2.$$

a) $E(B) = \frac{10}{3}$ hour, $\text{var}(B) = \frac{1}{12}\left(\frac{20}{3}\right)^2 = \frac{100}{27} = 3.7$ hour^2. Further, $E(R) = \frac{20}{9}$ hour, and $\rho = \lambda E(B) = \frac{5}{6}$.

b) We have

$$\begin{aligned} E(W) &= E(L^q) \cdot E(B) + \rho \cdot E(R) + (1-\rho) \cdot \frac{1}{1+e^{-1/4} \cdot 4} \cdot \frac{1}{2}, \\ E(L^q) &= \lambda \cdot E(W). \end{aligned}$$

Hence

$$E(W) = \frac{\rho}{1-\rho} \cdot E(R) + \frac{1}{1+e^{-1/4} \cdot 4} \cdot \frac{1}{2} = \frac{100}{9} + \frac{1}{1+e^{-1/4} \cdot 4} \cdot \frac{1}{2} = 11.23 \text{ hour}$$

and

$$E(S) = E(W) + E(B) = 14.56 \text{ hour}.$$

c) Now we have $B_1 = U[0, \frac{10}{3}]$, $B_2 = U[\frac{10}{3}, \frac{20}{3}]$, $\lambda_1 = \lambda_2 = \frac{1}{8}$. So $E(B_1) = \frac{5}{3}$ hour, $E(B_2) = 5$ hour, $\rho_1 = \frac{5}{24}$ and $\rho_2 = \frac{5}{8}$. For the small (high priority) orders we get

$$\begin{aligned} E(W_1) &= E(L_1^q) \cdot E(B_1) + \rho \cdot E(R) + (1-\rho) \cdot \frac{1}{1 + e^{-1/4} \cdot 4} \cdot \frac{1}{2}, \\ E(L_1^q) &= \lambda_1 \cdot E(W_1), \end{aligned}$$

so

$$E(W_1) = \frac{\rho}{1-\rho_1} \cdot E(R) + \frac{1-\rho}{1-\rho_1} \cdot \frac{1}{1 + e^{-1/4} \cdot 4} \cdot \frac{1}{2} = 2.36 \text{ hour}$$

and

$$E(S_1) = E(W_1) + E(B_1) = 4.03 \text{ hour.}$$

For the large (low priority) orders, it follows that

$$\begin{aligned} E(W_2) &= E(L_1^q) \cdot E(B_1) + E(L_2^q) \cdot E(B_2) + \rho \cdot E(R) + (1-\rho) \cdot \frac{1}{1 + e^{-1/4} \cdot 4} \cdot \frac{1}{2} \\ &\quad + \lambda_1 \cdot E(W_2) \cdot E(B_1), \\ E(L_2^q) &= \lambda_2 \cdot E(W_2), \end{aligned}$$

so

$$E(W_2) = \frac{E(W_1)}{1-\rho} = 14.16 \text{ hour}$$

and

$$E(S_2) = E(W_2) + E(B_2) = 19.16 \text{ hour.}$$
