Exam Queueing Theory
Monday, May 19, 2014, 13.30 – 16.30 hour.

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1. Jobs arrive according to a Poisson stream with a rate of 2 jobs per hour at a manufacturing system consisting of two machines, labeled 1 and 2. Both machines work with speed 1. Each job is, independent of all other jobs, with probability $\frac{3}{4}$ of type 1 and with probability $\frac{1}{4}$ of type 2. The processing time of a type 1 job is exponential with a mean of $\frac{1}{2}$ hours. The processing time of a type 2 job is exponential with a mean of 1 hour. On arrival, type 1 jobs are sent to machine 1 and jobs of type 2 are sent to machine 2. Both machines process jobs in order of arrival.

   a) Argue that, at each machine, jobs arrive according to a Poisson stream, and that the number of jobs at machine 1 and the number of jobs at machine 2 are independent of each other.

   b) Give the joint equilibrium probabilities $p_{i,j}$ that there are i jobs ($i = 0, 1, 2, \ldots$) at machine 1 and j jobs ($j = 0, 1, 2, \ldots$) at machine 2.

   c) Determine the mean waiting time and the probability that the duration of the waiting time of an arbitrary job exceeds 1 hour.

   A period during which there are 2 or more jobs in the whole system (i.e., the jobs at the two machines together) is called "crowded."

   d) Determine the mean number of crowded periods per day (i.e., 24 hours).

   e) Determine the expected duration of a crowded period.

2. Consider the same arrival process of jobs as in exercise 1. However, the two (slow) machines are now replaced by a single fast machine, that works with speed 2. All jobs are sent to this fast machine, and the jobs are processed in order of arrival.

   a) Show that the Laplace-Stieltjes transform of the processing time of an arbitrary job on this fast machine in hours is given by

   $$\widetilde{B}(s) = \frac{8 + \frac{7}{2}s}{(4+s)(2+s)}.$$

   b) Show that the Laplace-Stieltjes transform of the waiting time in hours of a job is given by

   $$\widetilde{W}(s) = \frac{3}{8} + \frac{9}{16} \cdot \frac{1}{1+s} + \frac{1}{16} \cdot \frac{3}{3+s}.$$

   c) Determine the probability that the duration of the waiting time of an arbitrary job exceeds 1 hour.

   d) What are the mean waiting time of a job and the mean number of jobs waiting in the queue?

3. The inter-arrival times of jobs at a single machine consist of two independent exponential phases: the first phase with a mean of 4 minutes and the second phase with a mean of 6 minutes. The production time of a job on the machine is exponentially distributed with a mean of 6 minutes.

   a) Determine the equilibrium distribution of the number of jobs in the system just before an arrival of a new job.

   b) Use this equilibrium distribution to determine the Laplace-Stieltjes transform of the waiting time. What is the probability that the duration of the waiting time exceeds 12 minutes?

   c) Determine the mean number of jobs in the system at arrival instants and at arbitrary instants.

4. A 3D printing system is processing customer orders in order of arrival. Orders arrive according to a Poisson stream with a rate of 2 orders per day (8 hours). Each order requires the printing of a (customer specific) product. This product is printed on a tray, and it is build layer by layer. It takes exactly 8 minutes to print a layer of 1 cm. The height of a tray required for an order is uniformly distributed between 0 and 50 cm. When the printer becomes idle, it is immediately cleaned. The time to clean the printer is exactly 1 hour. After cleaning, the printer starts printing again, if there are orders waiting, and otherwise, it waits for the first order to arrive.

   a) Determine the mean and variance of the time to print a customer order.

   b) Determine the mean flow time (waiting time plus printing time) of a customer order.

   c) Orders that require to print a tray, the height of which is less than 25 cm, are given priority over the other orders. Interruption of printing is not allowed. Determine the mean flow time of the high priority orders and low priority orders.

Credits:

1a	b	c	d	e	2a	b	c	d	3a	b	c	4a	b	c
1	2	3	2	2	2	4	2	2	3	4	3	3	3	4