Answers Exam Queueing Theory
Monday, June 16, 2014, 15.00 – 18.00 hour.

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1. a) We have $p_0 = \frac{110}{371}$, $p_1 = \frac{99}{371}$ and $p_{n+2} = \left(\frac{9}{20}\right)^n \cdot \left(\frac{9}{10}\right)^2 \cdot \frac{110}{371}$ for $n = 0, 1, 2, \ldots$.

   b) $E(L) = \sum_{n=0}^{\infty} n\, p_n \approx 1.5$ and hence $E(S) = E(L)/\lambda \approx 1/6$ hour $= 10$ minutes. The system in which the second machine is never used is an $M/M/1$ queue with arrival rate $\lambda = 9$ and service rate $\mu = 10$. For this system $E(S) = 1/(\mu - \lambda) = 1$ hour $=$ 60 minutes. Hence, we have a reduction of 50 minutes.

   c) $\lambda\, p_2 = 9\, p_2 \approx 2.16$ times per hour.

   d) The duration of a period during which both machines are simultaneously processing jobs behaves like a busy period in an $M/M/1$ queue with arrival rate $\lambda = 9$ and service rate $\mu = 20$. For this system we have $E(BP) = 1/(\mu - \lambda) = 1/11$ hours $=$ 5.45 minutes.

2. a)
$$\widetilde{B}(s) = E(e^{-sB}) = \frac{2}{5}\frac{4}{4+s} + \frac{3}{5}\left(\frac{4}{4+s}\right)^2 = \frac{80+8s}{5(4+s)^2}\,.$$
   Furthermore via the relation
$$\widetilde{R}(s) = \frac{1-\widetilde{B}(s)}{sE(B)}$$
   or directly via
$$\widetilde{R}(s) = \frac{5}{8}\frac{4}{4+s} + \frac{3}{8}\left(\frac{4}{4+s}\right)^2$$
   we obtain that
$$\widetilde{R}(s) = \frac{32+5s}{2(4+s)^2}.$$

   b) We have $\lambda = 5/3$ truck per hour, $E(B) = 2/5$ hours and $\rho = \lambda E(B) = 2/3$. Hence
$$\begin{aligned}\widetilde{W}(s) &= \frac{(1-\rho)s}{\lambda\widetilde{B}(s) + s - \lambda} \\ &= \frac{\frac{1}{3}s}{\frac{5}{3}\frac{80+8s}{5(4+s)^2} + s - \frac{5}{3}} \\ &= \frac{s^2+8s+16}{3s^2+19s+16} \\ &= \frac{1}{3} + \frac{9}{13}\frac{1}{1+s} - \frac{1}{39}\frac{16}{16+3s}\end{aligned}$$

   c) From b) we have
$$E(W) = \frac{1}{3}\cdot 0 + \frac{9}{13}\cdot 1 - \frac{1}{39}\cdot\frac{3}{16} = \frac{11}{16}.$$
   Hence,
$$E(S) = E(W) + E(B) = \frac{87}{80}$$

and

$$E(L) = \lambda E(S) = \frac{29}{16}.$$

3. Time unit: minute.

   We have $\lambda_1 = \frac{1}{6}, E(B_1) = 4, E(R_1) = 3, \rho_1 = \frac{2}{3}, \lambda_2 = \frac{1}{20}, E(B_2) = E(R_2) = 5, \rho_2 = \frac{1}{4}$.

   a) $$E(S_1) = E(B_1) + \frac{\rho_1 E(R_1)}{1-\rho_1} = 10 \text{ minutes.}$$

   b) $$E(S_1) = E(B_1) + \frac{\rho_1 E(R_1) + \rho_2 E(R_2)}{1-\rho_1} = 13.75 \text{ minutes.}$$

   c) $$E(S_1) = E(B_1) + \frac{\rho_1 E(R_1)}{1-\rho_1} = 10 \text{ minutes.}$$

   d) $$E(S_1) = E(B_1) + \frac{\rho_1 E(R_1) + (1-\rho_1) E(R_2)}{1-\rho_1} = 15 \text{ minutes.}$$

4. The arrival rate is $\lambda = 1$ per hour and the mean processing time is $E(B) = \frac{5}{6}$ with a standard deviation of $\sigma(B) = \frac{5}{12}$ hour. Hence the coefficient of variation is $c_B = \frac{1}{2}$ and the mean residual residual processing time is $E(R) = \frac{25}{48}$ hour.

   a) The utilization $\rho = \frac{5}{6}$, and

   $$E(W) = \frac{\rho}{1-\rho} \cdot E(R) = \frac{125}{48} = 2.604 \text{ hour.}$$

   b) Let S be the exponential setup time, so $E(S) = \frac{1}{2}$. Then

   $$E(W) = \frac{\rho}{1-\rho} \cdot E(R) + E(S) = \frac{149}{48} = 3.104 \text{ hour.}$$

   c) The reduction in power costs per hour is

   $$120 \cdot (1-\rho) \cdot \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + E(S)} = \frac{40}{3} = 13.3 \text{ euros per hour.}$$

   d) Let T denote the fixed waiting time before switching off the machine, so $T = \frac{1}{2}$ hour. Then

   $$E(W) = \frac{\rho}{1-\rho} \cdot E(R) + \frac{e^{-\lambda T}\left(\frac{1}{\lambda} + E(S)\right)}{\frac{1}{\lambda} + e^{-\lambda T} E(S)} \cdot E(S) = \frac{125}{48} + \frac{\frac{3}{4} e^{-\frac{1}{2}}}{1 + e^{-\frac{1}{2}} \frac{1}{2}} = 2.95 \text{ hour.}$$

   e) The reduction in power costs per hour is

   $$120 \cdot (1-\rho) \cdot \frac{e^{-\lambda T} \frac{1}{\lambda}}{\frac{1}{\lambda} + e^{-\lambda T} E(S)} = \frac{20 e^{-\frac{1}{2}}}{1 + e^{-\frac{1}{2}} \frac{1}{2}} = 9.31 \text{ euros per hour.}$$