Exam for LNMB/Mastermath Course on Scheduling
01 June 2015

This exam consists of:

- **2 pages.**
- **5 questions.**
- You can obtain a total of 50 points. Your exam grade will be the points you obtained divided by 5.

When a proof is asked, please provide a mathematically sound proof, short but precise. Unless stated otherwise, you are always expected to (briefly) explain your answer.

In case the objective function is not explicitely specified, it is supposed to be a regular objective function, unless stated otherwise.

Exercise 1 (10 points)
Consider the problem $1 \,|\,|\, \sum w_j U_j$. Show that this problem is NP-hard by a reduction from Knapsack.

In the (decision version of the) Knapsack problem, q items with values $v_1, \ldots, v_q$ and sizes $s_1, \ldots, s_q$ are given as well as a knapsack with capacity B and a bound V. The question is whether there exists a subset of the items $S \subset \{1, \ldots, q\}$ such that the value $\sum_{i \in S} v_i$ of items in the knapsack is at least V subject to the constraint that the total size of these items is no more than the capacity, i.e., $\sum_{i \in S} s_i \leq B$.

Exercise 2 (10 points)
Design a dynamic programming algorithm that solves $Q2 \,|\,|\, \sum w_j C_j$ to optimality. Assume that $s_1 = 1$ and $s_2 = s \geq 1$.

What is the running time of your algorithm?

Exercise 3 (10 points)
Consider the problem $P \,|\,|\, C_{\max}$ and the following scheduling rule. Start with the empty schedule. While not all jobs have been assigned, select an unscheduled job and schedule it at the end of the currently least loaded machine.

Show that this algorithm is a $(2 - \frac{1}{m})$-approximation.

Note: You can also show that it is a 2-approximation, which of course will not give you the full points.

Exercise 4 (10 points)
Consider the problem J$\,|\,|\,C_{\max}$ and its disjunctive graph formulation. Let S be a complete selection and let P be a critical path in $G(S)$, i.e., a longest path from $\circ$ to $\star$ in $G(S)$.

(a) Show that reversing the direction of a disjunctive arc which is part of the critical path P leads again to a complete selection.

(b) Show that this is in general not true for an arbitrary disjunctive arc.

Exercise 5 (10 points)
Consider the stochastic scheduling problem P$\,|\,|\,\mathbb{E}\,[\,C_{\max}\,]$ with three jobs, having the following processing time distributions:

- Job 1 has deterministic processing time of 4.
- Job 2 has processing time Pr $[P_2 = 1] = \frac{3}{12}$, Pr $[P_2 = 3] = \frac{7}{12}$ or Pr $[P_2 = 6] = \frac{2}{12}$. Note that $\mathbb{E}\,[\,P_2\,] = 3$.
- Job 3 is (continuous) uniformly distributed over $[2, 5]$. Note that $\mathbb{E}\,[\,P_3\,] = \frac{7}{2} = 3.5$.

(a) Determine an optimal policy for the case of $m = 1$.

(b) Determine an optimal policy for the case of $m = 2$.

(c) Determine an optimal policy for the case of $m = 3$.

In all three case, also determine the expected makespan.