Test $T_1$, "Scientific Computing", 22-02-2012

1. (25 points[1]) Determine $L \in \mathbb{R}^{3\times 3}$ such that $L^{-1} = L_2 L_1$,

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}, \quad a, b, c \in \mathbb{R}.$$

You are supposed to write down $L$, rather than to compute $L$.

$$L = \underbrace{(L_2 L_1)^{-1} = L_1^{-1} L_2^{-1}}_{5p.} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}}_{10p.} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & -c & 1 \end{bmatrix}}_{10p.}$$

2. (10 p) Give the definition of the Schur decomposition.

For any $A \in \mathbb{C}^{n\times n}$ there exist unitary $Q \in \mathbb{C}^{n\times n}$ and upper triangular $T \in \mathbb{C}^{n\times n}$ such that

$Q^*AQ = T \overset{\text{or}}{\Leftrightarrow} A = QTQ^*$. 5p.

Diagonal entries of $T$ are the e.values of $A$. $Q$ can be chosen such that the e.values appear on the main diagonal of $T$ in any chosen order 5p.

3. (25 p) Determine the Schur decomposition of the matrix $A$ (given below) by finding suitable permutations of rows and columns.

$$A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}, \quad a, b, c, d, e, f \in \mathbb{R}.$$

Let $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then $PA = \begin{bmatrix} d & e & f \\ b & c & 0 \\ a & 0 & 0 \end{bmatrix}$ and

$$PAP = \underbrace{\begin{bmatrix} f & e & d \\ 0 & c & b \\ 0 & 0 & a \end{bmatrix}}_{T \text{ upper triangular}} \text{ and } P = P^T = P^*$$

Answer: $PAP^* = T$ with

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } T = \begin{bmatrix} f & e & d \\ 0 & c & b \\ 0 & 0 & a \end{bmatrix}$$

[1] Total number of points is 100.

4. (10 p) Give the definition of the SVD of a matrix $A \in \mathbb{C}^{m\times n}$, $m \geqslant n$.

For any $A \in \mathbb{C}^{m\times n}$ there exist unitary $\underset{m\times m}{U}$ and $\underset{n\times n}{V}$ suchthat

$$A = U\Sigma V^* \overset{\text{or}}{\Longleftrightarrow} U^*AV = \Sigma = \mathrm{diag}\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{m\times n}$$

with $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_n \geqslant 0$.

5. (30 p) Let $A \in \mathbb{R}^{3\times 2}$. It is known that if a vector $x \in \mathbb{R}^2$ is written as $x = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, with $\alpha, \beta \in \mathbb{R}$, then

$$Ax = 2\alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Determine the SVD of $A$. Motivate your answer.

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent. Hence any $x \in \mathbb{R}^2$ can be written as their linear combination. Moreover, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are orthogonal.

$$Ax = A\left(\alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = A\left(\alpha\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + \beta\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right) =$$

$$= \sqrt{2}\, A \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{\text{denote by } V,\ V \text{ is unitary}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} =$$

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add any vector which is orthonormal to the first two columns

10p.

$$= 2\alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\alpha \\ \beta \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{unitary, denote by } U} \begin{bmatrix} 2\alpha \\ \beta \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\Sigma\,?} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

10p

Since $\alpha$ and $\beta$ are any numbers, we have $AV = \frac{1}{\sqrt{2}} U\Sigma$

This seems to be the SVD except for factor $\sqrt{2}$ (unitary), but we can bring it into $\Sigma$:

$$AV = U \underbrace{\begin{bmatrix} 2/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}}_{\text{this is } \Sigma\,!}$$

10p.