Scicomp T1, 17-02-2015, Solution

1. (10 points[1]) Give a definition of an LU factorization with partial pivoting, indicating for which matrices it exists.

For any nonsingular $\underset{n\times n}{A}$ there exist P permutation, L lower triang., U upper triang. matrices such that $PA = LU$

2. (30 p) For the matrix A given below, carry out the first step of the LU factorization with partial pivoting, which involves a permutation matrix P_1 and an elementary row operation matrix L_1. Specify P_1, L_1 and $L_1 P_1 A$.

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 2 & -5 & 6 & -7 \\ -3 & 0 & 8 & -9 \\ 4 & 0 & 0 & 10 \end{bmatrix}$$

$$P_1 A = \begin{bmatrix} 4 & 0 & 0 & 10 \\ 2 & -5 & 6 & -7 \\ -3 & 0 & 8 & -9 \\ 1 & 2 & -3 & 4 \end{bmatrix} \quad \text{with } P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$L_1 P_1 A = \begin{bmatrix} 4 & 0 & 0 & 10 \\ 0 & -5 & 6 & -12 \\ 0 & 0 & 8 & -1.5 \\ 0 & 2 & -3 & 1.5 \end{bmatrix} \quad \text{with } L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 3/4 & 0 & 1 & 0 \\ -1/4 & 0 & 0 & 1 \end{bmatrix}$$

3. (15 p) For given vectors $x, y \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n\times n}$ the following product has to be computed:

$$y^T(AB)x$$

Should the brackets in the product be placed differently to minimize the computational work? Give a short explanation.

AB costs $\approx O(n^3)$

$(y^T A)(Bx)$ is better : $\approx O(n^2)$

[1]The total number of points is 90 and the grade is determined as $G = 1 + P/10$ where P is the number of points earned.

4. (10 p) Give a definition of an SVD indicating for which matrices it exists.

$m \times m$ $n \times n$

for any $A \in \mathbb{C}^{m \times n}$ there exist U, V unitary matrices and diagonal Σ $(m \times n)$ with nonnegative entries $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p$, $p = \min(m,n)$:

$A = U \Sigma V^*$

5. (15 p) A matrix A has an SVD $U\Sigma V^T$ with

$$\Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2}. \end{bmatrix}$$

What is the null space of A? Give a short explanation.

$A = U \Sigma V^*$ solve $Ax = 0$

$U \Sigma V^* x = 0$ and let $V^* x = y$

$U \Sigma y = 0$

$\Sigma y = U^* 0 = 0 \Leftrightarrow y = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}, \alpha \in \mathbb{C}$

$\Leftrightarrow x = Vy = \alpha \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \Leftrightarrow \text{nul}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

6. (10 p) For the matrix A from Question 5, determine $\|A\|_2$. Provide a short explanation.

$\|A\|_2 = \sigma_1$ = largest sing. value $= 3$