Solutions T_2, 25-04-2012

Question 1 (30 points) Let $A \in \mathbb{R}^{n\times n}$, $r \in \mathbb{R}^n$ and $m \ll n$ be given. What can you say about the following algorithm and the matrices $V_{m+1} \in \mathbb{R}^{n\times(m+1)}$ and $B \in \mathbb{R}^{(m+1)\times m}$ produced by it? Here, v_j denotes column j of V_{m+1}.

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1.  define zero matrices B ∈ R^{(m+1)×m} and V ∈ R^{n×(m+1)}
2.  β := ||r||_2, v_1 := r/β
3.  for j = 1,2,...,m do
4.      w := A v_j
5.      for i = 1,2,...,j do
6.          b_ij := (w, v_i)
7.      endfor
8.          w := w − Σ_{i=1}^{j} b_ij v_i
9.      b_{j+1,j} := ||w||_2
10.     if b_{j+1,j} = 0 stop
11.     v_{j+1} := w/b_{j+1,j}
12. endfor
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(1) It is the Arnoldi process because
(a) Krylov subspace is built up — 8p
(b) vectors v_j orthogonalized by <u>classical</u> Gram-Schmidt — 7p

Hence, Arnoldi relation holds:

$$A V_m = V_{m+1} B$$

$V_{m+1}^{*} V_{m+1} = I$ identity $(m+1)\times(m+1)$ — 7p

B upper Hessenberg

provided that the process is not stopped at step 10. — 8p

Question 2 (30 points) Assume a linear system $Ax = b$ is being solved for given $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. An iterative method for its solution has a property that its residual r_k after k iterations satisfies $r_k = P_k(A)r_0$, where $r_0 = b - Ax_0$ is the initial residual, x_0 is the initial guess and P_k is a polynomial of degree k. Is this iterative method a Krylov subspace method? Explain why or why not.

It is a Krylov subspace method provided

$$x_k = x_0 + z_k, \quad z_k \in \mathcal{K}_k(A, r_0) \iff z_k = Q_{k-1}(A) r_0$$

Then for the residual would hold — 10p

$$r_k = b - Ax_k = b - Ax_0 - Az_k = r_0 - AQ_{k-1}(A) r_0$$

$$= \underbrace{\left(I - AQ_{k-1}(A)\right)}_{\text{Residual polynomial } P_k\text{, with property } P_k(0)=1} r_0$$

10p.

Answer: it is <u>not</u> a Krylov subspace method

It would be if we had: $P_k(0) = 1$.

10p.

Question 3 (20 points) Assume m steps of the Arnoldi process for a matrix $A \in \mathbb{R}^{n\times n}$ and vector $r \in \mathbb{R}^n$ produced the matrices $V_{m+1} \in \mathbb{R}^{n\times(m+1)}$ and $\underline{H}_m \in \mathbb{R}^{(m+1)\times m}$, $m < n$ Let H_m be formed by the first m rows of $\underline{H}_m$.

(a) Give a definition of the Rayleigh quotients of A. Prove that the Rayleigh quotients of H_m form a subset of the Rayleigh quotients of A.

(b) Assume now that $A = I + S$ where I is the identity matrix and $S = -S^*$. Prove that the Rayleigh quotients of H_m lie in the complex plane on the line $\{z \in \mathbb{C}\,|\, z = 1 + iy,\ y \in \mathbb{R}, i^2 = -1\}$.

(a) Rayleigh quotients $= \left\{ \frac{(Ax,x)}{(x,x)},\ x \neq 0 \right\} \longleftarrow \supset \longleftarrow$ 10p

$$\left\{ \frac{(H_m x, x)}{(x,x)},\ x \neq 0 \right\} = \left\{ \frac{(V_m^* A V_m x, x)}{(V_m^* V_m x, x)} \,\middle|\, x \neq 0 \right\} = \left\{ \frac{(A(V_m x), (V_m x))}{((V_m x), (V_m x))} \,\middle|\, x \neq 0 \right\}$$

(b) $(Ax, x) = (x,x) + (Sx, x)$

$z = (Sx, x) = +(x, S^*x) = -(x, Sx) = -\overline{(Sx, x)} = -\bar{z} \Leftrightarrow$ 10p

z is purely imaginary. Hence R. quotients of A and H_m are $1 + iy$, $y \in \mathbb{R}$

Question 4 (20 points) Let $A \in \mathbb{R}^{n\times n}$, $g(t) : \mathbb{R} \leftarrow \mathbb{R}^n$ and $w^0 \in \mathbb{R}^n$ be given. The BDF2 scheme for the time integration of the IVP $w'(t) = -Aw(t) + g(t)$, $w(0) = w^0$ reads:

$$\frac{3}{2} w^{k+2} - 2w^{k+1} + \frac{1}{2} w^k = -\tau A w^{k+2} + g^{k+2}, \quad k \geqslant 0,$$

where w^k is the approximate solution at time $t_k = k\tau$, $w^k \approx w(t_k)$, $\tau > 0$ is the time step size and $g^k = g(t_k)$. Assume also that w^1 is known.

(a) Rewrite BDF2 in the form of a linear system that has to be solved at every time step.

(b) Assume $A = A^T$. Write this linear system in a preconditioned form, such that an application of the MINRES iterative solver would be possible.

(a) $\underbrace{\left(\frac{3}{2} I + \tau A\right)}_{\hat{A}} w^{k+2} = b \underbrace{\left(= g^{k+2} + 2w^{k+1} - \frac{1}{2} w^k\right)}_{\text{known terms, right-hand side}}$ 6p

(b) $\underbrace{M_1^{-1} \hat{A} M_2^{-1}}_{\tilde{A}} \underbrace{M_2 w^{k+2}}_{\tilde{w}^{k+2}} = \underbrace{M_1^{-1} b}_{\tilde{b}}$, M1^T = M2 6p

Two sided preconditioner because otherwise symmetry of $\hat{A}$ can be lost and application of MINRES would not be possible. 8p