T_2 24-04-2013

Question 1 (40 points) A linear system $Ax = b$ is solved, with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ given.

(a) (10 p) Write down the right-preconditioned system $\tilde{A}\tilde{x} = \tilde{b}$ for a preconditioner matrix $M \in \mathbb{R}^{n\times n}$—more precisely, specify $\tilde{A}$, $\tilde{x}$, $\tilde{b}$ in terms of A, x, b and M.

$$Ax = b \rightarrow \underbrace{AM^{-1}}_{\tilde{A}}\underbrace{Mx}_{\tilde{x}} = \underbrace{b}_{\tilde{b}}$$

(b) (15 p) Write down the (unpreconditioned) Richardson method for solving the right-preconditioned system given above. After that rewrite the Richardson method in terms of A, x, b and M.

$$\tilde{x}_{k+1} = \tilde{x}_k + \tilde{r}_k \quad \text{or} \quad \tilde{x}_{k+1} = \tilde{x}_k + \tilde{b} - \tilde{A}\tilde{x}_k$$

unprec. Richardson

$$\tilde{x}_{k+1} = \tilde{x}_k + \tilde{b} - \tilde{A}\tilde{x}_k$$

$$M x_{k+1} = M x_k + b - AM^{-1}Mx_k$$

$$Mx_{k+1} = Mx_k + b - Ax_k$$

$$x_{k+1} = x_k + M^{-1}\underbrace{(b - Ax_k)}_{r_k}$$

(c) (15 p) For which choice of M will the Richardson iteration converge in the fastest possible way? Motivate your answer.

For $M = A$ convergence in one iteration:

given x_0 and $r_0 = b - Ax_0$

$$x_1 = x_0 + M^{-1}r_0 = x_0 + A^{-1}r_0 =$$

$$= x_0 + A^{-1}(b - Ax_0) = x_0 + A^{-1}b - x_0 = \underbrace{A^{-1}b}_{\text{exact solution}}$$

Question 2 (30 points) For a matrix $A \in \mathbb{R}^{n\times n}$ it is known that all its Rayleigh quotients lie on the line $2 + i\beta$ in the complex plane, with $\beta \in \mathbb{R}$ and $i^2 = -1$. The line is thus parallel to the imaginary axis and crosses the real axis at point $2 + i0 = 2$. Is it true that the Ritz values of A will also lie on the line $2 + i\beta$? Why or why not?

Ritz values of A are the eigenvalues of $H_m = V_m^* A V_m$

$$\text{Rayleigh quot.} = \left\{ \frac{(Ax,x)}{(x,x)} \,\middle|\, x \neq 0 \right\} \supset \left\{ \frac{(Ax,x)}{(x,x)} \,\middle|\, \underset{x\neq 0}{x = V_m y},\, y \in \mathbb{C}^m \right\} =$$

$$= \left\{ \frac{(AV_m y, V_m y)}{(V_m y, V_m y)} \,\middle|\, y \neq 0 \right\} = \left\{ \frac{(V_m^* A V_m y, y)}{(V_m^* V_m y, y)} \,\middle|\, y \neq 0 \right\} =$$

$$= \left\{ \frac{(H_m y, y)}{(y,y)} \,\middle|\, y \neq 0 \right\} = \text{Rayleigh quot. of } H_m \supset \text{eigenvalues of } H_m$$

Answer: Yes

Question 3 (30 points) Write down the implicit trapezoidal rule for the initial value problem $w'(t) = -Aw(t) + g(t)$, $w(0) = w^0$. After that rewrite the scheme as a linear system where the unknown vector is the solution w^{k+1} at the next time level, i.e., $w^{k+1} \approx w(\tau(k+1))$, with $\tau > 0$ being the time step size and k the time step index.

ITR: $$\frac{w^{k+1} - w^k}{\tau} = -\frac{1}{2} A w^k - \frac{1}{2} A w^{k+1} + \frac{1}{2}(g^k + g^{k+1})$$

$$w^{k+1} - w^k = -\frac{\tau}{2} A w^k - \frac{\tau}{2} A w^{k+1} + \frac{\tau}{2}(g^k + g^{k+1})$$

$$\underbrace{\left(I + \frac{\tau}{2} A\right)}_{\hat{A}} w^{k+1} = \underbrace{\left(I - \frac{\tau}{2} A\right) w^k + \frac{\tau}{2}(g^k + g^{k+1})}_{\hat{b}}$$

linear system for unknown w^{k+1}