T_3 01.06.12. Solutions

Question 1 (35 points) Sylvester equation $AX - XB = C$ is solved for given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{k\times k}$, $C \in \mathbb{R}^{n\times k}$ and unknown $X \in \mathbb{R}^{n\times k}$.

Q1a (10 p) Specify $(\mathrm{vec}(X))^T$ in terms of its columns x_i, $i = 1, \ldots, k$:
$(\mathrm{vec}(X))^T = \left(x_1^T \ \ x_2^T \ \ \ldots \ \ x_k^T \right)$

Q1b (10 p) The Sylvester equation is transformed into an equivalent linear system $\mathcal{A}\,\mathrm{vec}(X) = \mathrm{vec}(C)$. Specify, without proof, the missing terms in the following formula (here I_n $\in$ $\mathbb{R}^{n\times n}$ is the identity matrix):

$$\mathcal{A} = I_k \otimes A - B^T \otimes I_n$$

, I_k is identity $k \times k$

Q1c (15 p) Write down the matrix $\mathcal{A}$ for A and B given below:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{A} = \left[\begin{array}{c|ccc} A & 1 & & \\ & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ \quad -1 & & A & \\ \qquad -1 & & & \end{array}\right]$$

Question 2 (30 points) A nonlinear system of equation $F(x) = 0$ is solved by an inexact Newton method, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth mapping.

Q2a (10 p) Complete the formula below for the matrix free multiplication of the Jacobian times a vector $w \in \mathbb{R}^n$ (here $\delta > 0$ is a small parameter and $x_c \in \mathbb{R}^n$ is the current solution vector):

$$F'(x_c)w \approx \frac{1}{\delta}(F(x_c + \delta w) - F(x_c)). \tag{1}$$

Q2b (10 p) Estimate accuracy of approximation (1) above, i.e. prove that the approximation error is $O(\delta^\ell)$ and specify ℓ.

$$F'(x_c)w - \frac{1}{\delta}\left(F(x_c+\delta w) - F(x_c)\right) = F'(x_c)w - \frac{1}{\delta}\left(F'(x_c)\,\delta w + O(\|\delta w\|^2)\right) =$$
$$= F'(x_c)w - F'(x_c)w + O(\delta) = O(\delta), \text{ thus } \ell = 1$$

Q2c (10 p) How could we improve the accuracy of approximation (1)?

$$F'(x_c)w \approx \frac{1}{2\delta}\left(\underbrace{F(x_c+\delta w) - F(x_c - \delta w)}_{2\delta F'(x_c)w + O(\delta^3)}\right)$$ – Central finite differences

$\Downarrow$

error $\frac{1}{\delta} O(\delta^3) = O(\delta^2)$

Question 3 (35 points) Consider the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update, for given $B_k \in \mathbb{R}^{n\times n}$ and vectors x_k, x_{k+1}, y_k:

$$B_{k+1} = B_k + \frac{y_k y_k^T}{y_k^T(x_{k+1}-x_k)} - \frac{B_k(x_{k+1}-x_k)(B_k(x_{k+1}-x_k))^T}{(x_{k+1}-x_k)^T B_k(x_{k+1}-x_k)}$$

Q3a (10 p) What is the rank of the matrix $y_k y_k^T$? Motivate your answer.

$y_k y_k^T = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} [y_1 \cdots y_n] =$ columns of this matrix are the same up to a constant factor $\Rightarrow$ rank $= 1$

Q3b (10 p) What is the rank of the matrix $B_{k+1} - B_k$? Motivate your answer.

2 matrices of rank 1 $\Rightarrow$ rank 2

Q3c (15 p) Simplify

$B_{k+1}(x_{k+1} - x_k) = B_k(x_{k+1} - x_k) + \frac{y_k y_k^T \cancel{(x_{k+1} - x_k)}}{y^T \cancel{(x_{k+1} - x_k)}} -$

$$- \frac{B_k(x_{k+1}-x_k)\cancel{(B_k(x_{k+1}-x_k))^T(x_{k+1}-x_k)}}{\cancel{(x_{k+1}-x_k)^T B_k(x_{k+1}-x_k)}} = y_k$$