Solutions of T3. 10-06-2014

Q1 Consider, for $F : \mathbb{R}^n \to \mathbb{R}^n$, the following problem: find $x \in \mathbb{R}^n$ such that

$$F(x) = 0. \tag{1}$$

(5 p) (a) Formulate, without proof, the fixed point iteration for this problem if the fixed point mapping is $K(x) = x - F(x)$.

$$x_{k+1} = K(x_k) = x_k - F(x_k)$$

(15 p) (b) Formulate, without proof, a condition which is sufficient for the fixed point iteration to converge locally.

$\|K'(x)\| < 1$ for all x sufficiently close to solution x_*

$K'(x)$ is the Jacobian of $K(x)$

(20 p) (c) Consider the problem (1)

$$F(x) = \frac{1}{3}x * x - \frac{3}{2}x, \tag{2}$$

where $*$ is the elementwise multiplication of the two vectors. For example if $x = (1, 2, 3)^T$ and $y = (2, 2, 1)^T$ then $x * y = (2, 4, 3)^T$. Consider the following two fixed point iterations:

iteration I: $K(x) = x - F(x)$, iteration II: $K(x) = x + F(x)$.

Which iteration would you prefer to find solutions x of problem (1),(2) such that $x \approx 0$? Of course, $x = 0$ is a solution but the iterative scheme does not know this.

I: $K'(x) = I - \frac{2}{3}xI + \frac{3}{2}I = \left(\frac{5}{2} - \frac{2}{3}x\right)I$, $\|K'(x)\| > 1$ for $x \approx 0$

II: $K'(x) = I + \frac{2}{3}xI - \frac{3}{2}I = \left(-\frac{1}{2} + \frac{2}{3}x\right)I$, $\|K'(x)\| < 1$ for $x \approx 0$

Hence, I would use iteration II.

(10 p) (d) [Make this exercise as last.] Propose a preconditioner matrix $M \in \mathbb{R}^{n \times n}$ such that the preconditioned nonlinear problem (1),(2) could be solved with both fixed point iterations given above.

$M = 5I$ and consider the preconditioned nonlinear problem $\underbrace{M^{-1}F(x) = 0}_{\tilde{F}(x) = 0} \Rightarrow \tilde{F}'(x) = \left(\frac{2}{15}x - \frac{3}{10}\right)I$

We see that $\|\tilde{K}'(x)\| > 1$ again for iteration I

To have iteration I converge we take $M = -I$ (but then iteration II will not converge).

$F'(x) = \frac{2}{3}xI - \frac{3}{2}I$ where I is the identity matrix

Conclusion: M which helps I and II simultaneously is not possible.

(20 p) Q2 Consider matrix

$$M = \begin{bmatrix} A & A & A \\ B & B & B \\ A & A & A \end{bmatrix} \in \mathbb{R}^{300\times300},$$

where the matrices $A, B \in \mathbb{R}^{100\times100}$ are given. Write down a formula which defines M with the help of the Kronecker product in terms of A, B and some 3x3 matrices.

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \otimes A + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes B$$

(10 p) Q3 (a) For sufficiently smooth $F : \mathbb{R}^n \to \mathbb{R}^n$ write down the first two terms of the Taylor series of $F(x + \delta w)$, $x, w \in \mathbb{R}^n$, $\delta \in \mathbb{R}$ around point x.

$$F(x+\delta w) = F(x) + F'(x)\delta w + \text{h.o.t.}$$

$$\Rightarrow \quad F'(x)w = \frac{1}{\delta}\left(F(x+\delta w) - F(x)\right)$$

(10 p) (b) Using the Taylor expansion write down an approximation for the Jacobian matrix-vector product $F'(x)w$.