T2. 21.04.15. Solutions

Question 1 (45 points) A linear system $Ax = b$ is solved, with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ given.

(a) (20 p) Consider the following iterative method for solving the linear system $Ax = b$:

$$\begin{aligned} x_1 &= x_0 + \alpha_0 r_0, \\ x_{k+1} &= x_k + \alpha_k r_k + \beta_k r_{k-1}, \quad k \geqslant 1, \end{aligned} \tag{1}$$

where r_k is the residual of the approximate solution x_k and α_k and β_k are real scalars chosen such that

$$r_1 \perp r_0, \qquad r_{k+1} \perp r_k \quad \text{and} \quad r_{k+1} \perp r_{k-1}, \quad k \geqslant 1. \tag{2}$$

The scalars α_k, β_k for $k \geqslant 1$ can be found as a solution of a system of two linear equations, i.e., a linear system with a matrix of size 2×2. Derive this linear system.

$$r_{k+1} = b - A(x_k + \alpha_k r_k + \beta_k r_{k-1}) = r_k - \alpha_k A r_k - \beta_k A r_{k-1}$$ 5p.

$$0 = (r_k, r_{k+1}) = (r_k, r_k) - \alpha_k (r_k, A r_k) - \beta_k (r_k, A r_{k-1})$$

$$0 = (r_{k-1}, r_{k+1}) = \underbrace{(r_{k-1}, r_k)}_{=0} - \alpha_k (r_{k-1}, A r_k) - \beta_k (r_{k-1}, A r_{k-1})$$ 10p.

Linear system:

$$\begin{bmatrix} (r_k, A r_k) & (r_k, A r_{k-1}) \\ (r_{k-1}, A r_k) & (r_{k-1}, A r_{k-1}) \end{bmatrix} \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} \|r_k\|^2 \\ 0 \end{bmatrix}$$ 5p.

(b) (15 p) Assume now that $A = A^T$ and the 2×2 system from the previous question has a nonsingular matrix. Taking into account that

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \qquad (\text{for} \quad ac - b^2 \neq 0),$$

provide an explicit expression for α_k and β_k, $k \geqslant 1$.

If $A = A^T$ then $(r_k, A r_{k-1}) = (A r_k, r_{k-1})$ and $(x,y) = (y,x)$ because everything is real. Then denote $a = (A r_k, r_k)$, $b = (A r_k, r_{k-1})$, $c = (A r_{k-1}, r_{k-1})$ and we have 5p

$$\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} \begin{bmatrix} \|r_k\|^2 \\ 0 \end{bmatrix} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \begin{bmatrix} \|r_k\|^2 \\ 0 \end{bmatrix} = \frac{\|r_k\|^2}{ac - b^2} \begin{bmatrix} c \\ -b \end{bmatrix} \Rightarrow$$ 5p

$$\alpha_k = \frac{\|r_k\|^2 (A r_{k-1}, r_{k-1})}{(A r_k, r_k)(A r_{k-1}, r_{k-1}) - (A r_k, r_{k-1})^2}, \quad \beta_k = \frac{-\|r_k\|^2 (A r_k, r_{k-1})}{(A r_k, r_k)(A r_{k-1}, r_{k-1}) - (A r_k, r_{k-1})^2}$$

5p.

(c) (10 p) Assume $A = A^T$. Compare (i.e., briefly discuss possible differences and similarities of) the CG method and the method given by (1).

If A is s.p.d. then the method given by (1) should be the CG because CG was built by the requirement that the residual is orthogonal w.r.t. the previous two residuals. 10p.

Question 2 (45 points) For given $A \in \mathbb{R}^{N \times N}$, $w^0 \in \mathbb{R}^N$ and $g(t) : \mathbb{R} \to \mathbb{R}^N$, consider initial-value problem (IVP)

$$w'(t) = -Aw(t) + g(t), \qquad w(0) = w^0,$$

where $w(t) : \mathbb{R} \to \mathbb{R}^N$ is the unknown vector function.

(a) (10 p) Write down the backward Euler method for solving the IVP, denoting the time step size by τ. After that rewrite the method in the form of a linear system where the unknown vector is the solution at the next time level w^{n+1}.

$$\frac{w^{n+1} - w^n}{\tau} = -Aw^{n+1} + g^{n+1} \qquad 5p.$$

$$(I + \tau A)\, w^{n+1} = \underbrace{w^n + \tau g^{n+1}}_{b} \qquad 5p$$

(b) (20 p) Let H and S be the Hermitian and skew-Hermitian parts of A, respectively. Let LL^T be the Cholesky factorization of the matrix $I + \tau H$ with I being the identity matrix. Assume a two-sided precondtioner with the preconditioner matrices $M_1 = L$ and $M_2 = L^T$ is applied to the linear system derived in the previous question. Specify the matrix of the preconditioned system in terms of I, τ, L and S. Give a relation between the w^{n+1} and the unknown vector $\widetilde{w}^{n+1}$ of the preconditioned system.

$$(I + \tau H + \tau S)\, w^{n+1} = b \Rightarrow L^{-1}(LL^T + \tau S)L^{-T} \underbrace{L^T w^{n+1}} = L^{-1} b \qquad 10p$$

$$\underbrace{(I + \tau L^{-1} S L^{-T})}_{\text{preconditioned matrix}}\, \tilde{w}^{n+1} = \tilde{b}$$

$$\tilde{w}^{n+1} = L^T w^{n+1} \qquad 10p.$$

(c) (15 p) Is it possible that the matrix of the preconditioned system from the previous system is the identity matrix plus a skew-symmetric matrix? Is it possible that the matrix H_k in the GMRES method applied to the preconditioned system is tridiagonal? Where in the complex plane are the Ritz values of the matrix of the preconditioned system located? Motivate your answers.

$$\tilde{S}^T = (L^{-1} S L^{-T})^T = (L^{-T})^T S^T L^{-T} = L^{-1}(-S)L^{-T} \rightarrow \text{skew symmetric.} \quad 5p.$$

$$H_k = V_k^T (I + \tau \tilde{S}) V_k = I + \tau \underbrace{V_k^T \tilde{S} V_k}_{\text{skew-symmetric}} \quad + \text{upper Hessenberg} \Rightarrow \text{tridiagonal!} \quad 5p.$$

Ritz values = e.values of H_k are on the line 5p.