T_2 Solution. 19-04-2016

5 p **Q1** We solve a linear system $Ax = b$ iteratively. Define the error and the residual vectors for an approximation $x_k \approx x$.

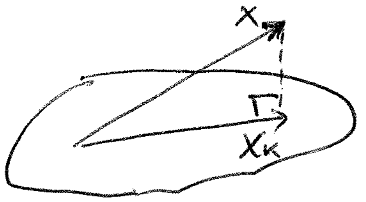
error $= x - x_k$, residual $= b - Ax_k$

10 p **Q2** Finish the definition of the FOM (fully orthogonal method):
An iterative method $x_k = x_0 + z_k$ for solving linear system $Ax = b$ is called FOM if the correction vector z_k is chosen in the Krylov subspace $\mathcal{K}_k(A, r_0)$ in such a way that ......

$r_k = b - Ax_k \perp \mathcal{K}_k(A, r_0)$, where $r_0 = b - Ax_0$.

20 p **Q3** For which A can we say that FOM minimizes the error? In which norm is the error minimized? Provide a short derivation.

For $A = A^*$ – in this case $r_k = Ax - Ax_k = A\underbrace{(x - x_k)}_{\text{error}}$



$\underbrace{(A(x-x_k), (x-x_k))}_{\text{in this norm the error is minimized}} = \|x - x_k\|_A^2$

15 p **Q4** We solve an eigenproblem $Ax = \lambda x$ where $A = BC^{-1}D$, the matrices B, C, D are large and sparse and given explicitly but A itself is not. Provide a short algorithmic description for a power method to solve this problem (indicating explicitly how approximate eigenvalue and eigenvectors are defined).

$x_0 =$ any vector, $x_0 \neq 0$

for $k = 1, \ldots$

$x_1 := Ax_0 \longleftarrow \begin{cases} y = Dx_0 \\ Cx_1 = y \leftarrow \text{solve} \\ x_1 = Bx_1 \end{cases}$

$x_1 := \dfrac{x_1}{\|x_1\|}$

$\lambda := (Ax_1, x_1) = x_1^* A x_1$

Same stopping criterion

end

Thus we avoid forming C^{-1} (full matrix $\to$ expensive)

10 p **Q5** Assume we solve an eigenproblem $Ax = \lambda x$. Finish the following statement: The shift-and-invert (SAI) acceleration of an iterative eigensolver means that, instead of the matrix-vector multiplications with the matrix A, the matrix-vector multiplications are carried out with the matrix ...... $(A - \sigma I)^{-1}$ ......

where ... σ is the shift $\approx$ the e.value we are searching for.

15 p **Q6** Can the power method as described in Question 4 be accelerated by the shift-and-invert (SAI) approach? If yes, which linear solver (direct, iterative or both) can be used for solving the SAI systems? Why?

To use the direct solve we would need to LU-factorize the matrix $A - \sigma I = BC^{-1}D - \sigma I$. For this, we would need the matrix C^{-1} explicitly. Therefore, only an iterative solver would be possible.

15 p **Q7** Recall that any Runge–Kutta method applied to the scalar test equation $w'(t) = \lambda w(t)$ can be written as $w^{n+1} = R(z)w^n$, $z = \tau\lambda$. Define the stability function and stability region of a Runge–Kutta method. Derive the stability function for the backward Euler method.

Stability function: $R(z)$

Stability region: $\{z \in \mathbb{C} \mid |R(z)| \leq 1\}$

Backward Euler:

$$\frac{w^{n+1} - w^n}{\tau} = \lambda w^{n+1}$$

$$w^{n+1} = w^n + z w^{n+1}$$

$$w^{n+1} = \underbrace{\frac{1}{1-z}}_{R(z) \text{ of backward Euler}} w^n$$

The grade is determined as $G = (10 + p)/10$, where p is the total number of points yearned.