Sci Comp exam 20-06-2016
Solution

T1.1 Matrices P and Q should be: $P = Q = I$

Furthermore,

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diagU = diag(U))
D = diag(sqrt(U));
all(diagU > 0)   - should be true
L = L*D;
...      Dinv = diag(1/sqrt(U));
U = Dinv*U;
norm(L-U',1)   - should be zero
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(5p)

Motivation: (1) A is pos. definite iff A has Cholesky factorization

(2) Cholesky is a particular sort of LU (3p)

T1.2 Algorithm $R_A(x)$ is called stable if

$$\frac{\|\delta x\|}{\|x\|} \text{ small } \Longrightarrow \frac{\|R_A(x+\delta x) - R_A(x)\|}{\|R_A(x)\|} \text{ small}$$

T1.3 A is called normal ~~provided that~~ if $AA^* = A^*A$.

$$\forall A \in \mathbb{C}^{n\times n} \; \exists \underset{\in \mathbb{C}^{n\times n}}{Q} \text{ unitary}, \; T \in \mathbb{C}^{n\times n} \text{ upper triangular}:$$

$$A = QTQ^*$$

where Q can be chosen such that the e.values of A appear in T (on the main diagonal) in any order.

T1.4 $AA^* = QTQ^*(QTQ^*)^* = QTQ^*QT^*Q^* =$
$= QTT^*Q^*$
$A^*A = \dots = QT^*TQ^*$ $\Big\} \Rightarrow TT^* = T^*T$ 5p.

Inspecting this equality elementwise shows that
T is diagonal. 5p.

(derivation should be given!)

T2.1 A method is called Krylov subspace method
if $x_k = \underset{\text{initial guess}}{x_0} + z_k, \quad z_k \in \mathcal{K}_k(A, r_0)$
$\mathcal{K}_k(A, r_0) = \text{span}\{r_0, Ar_0, \dots, A^{k-1}r_0\}$ –
– Krylov subspace, $r_0 = b - Ax_0$ 3p.

Galerkin condition: $r_k \perp \mathcal{K}_k(A, r_0)$ 3p.

T2.2 $x_1 = x_0 + z_1, \quad z_1 \in \mathcal{K}_1(A, r_0) = \text{span}\{r_0\}$
$\Rightarrow z_1 = \alpha r_0$ 5p.
$r_1 = b - Ax_1 = b - Ax_0 - Az_1 = r_0 - \alpha A r_0 \perp r_0$
$0 = (r_1, r_0) = (r_0, r_0) - \alpha (Ar_0, r_0) \Rightarrow \alpha = \dfrac{\|r_0\|^2}{(Ar_0, r_0)}$ 5p.

T2.3 Let $H = \frac{1}{2}(A + A^T) = LL^T$ (Cholesky)
Two-sided preconditioner $M = M_1 M_2 = LL^T$: 4p.
$\underbrace{L^{-1} A L^{-T}}_{\tilde{A}} L^T x = L^{-1} b$
$\tilde{A} = L^{-1}(H + S)L^{-T} = I + \underbrace{L^{-1}SL^{-T}}$ because $(L^{-1}SL^{-T})^T = -L^{-1}SL^{-T}$

S: skew-Herm. part of A
I: Hermitian part of $\tilde{A}$
$L^{-1}SL^{-T}$: Skew-Hermitian part of $\tilde{A}$ 4p.

T2.4 $y^{k+1}-y^{k} = -\frac{\tau}{2}Ay^{k+1} - \frac{\tau}{2}Ay^{k} + g(t_{k+1/2})$, $\quad t_{k+1/2} = (k+1/2)\tau$

$(I+\frac{\tau}{2}A)y^{k+1} = (I-\frac{\tau}{2}A)y^{k} + g^{k+1/2}$ 2p.

$(I+\frac{\tau}{2}A)(y^{k+1}-y^{k}) = y^{k} - \frac{\tau}{2}Ay^{k} - y^{k} - \frac{\tau}{2}Ay^{k} + g^{k+1/2}$

$(I+\frac{\tau}{2}A)\,x = -\tau A y_k + g^{k+1/2}$ 4p.

T3.1 The Sylvester equation has ! solution iff

(*) $\quad \Lambda(A) \cap \Lambda(B) = \emptyset$ 2p.

E.values of A lie on the imag. axis in $\mathbb{C}$

E.values of B lie on the real axis in $\mathbb{C}$. 4p.

(*) holds iff B is nonsingular. 4p.

T3.2 Newton method: $\quad x_{k+1} = x_k - (f''(x_k))^{-1}\nabla f(x_k)$

Secant method:
(for scalar problem) $\quad x_{k+1} = x_k - \dfrac{x_k - x_{k-1}}{\nabla f(x_k) - \nabla f(x_{k-1})}\nabla f(x_k)$

Secant method in general case:

$$x_{k+1} = x_k - B_k^{-1}\nabla f(x_k)$$ 5p.

where $\quad B_k(x_k - x_{k-1}) = \nabla f(x_k) - \nabla f(x_{k-1})$ 5p.
(quasi-Newton condition)

T3.3 (a) $\quad M^{-1}F(x) = 0, \quad \tilde{F}(x) = M^{-1}F(x)$ 5p

(b) $\quad x_{k+1} = x_k - M^{-1}F(x_k)$

In early iterations: $M = I$ (fixed point) or another fixed matrix

In subsequent iterations: $M = F'(x_k)$ (Newton method) 5p.