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Exam for the Course

Scientific Computing

Quartile 4 2017/2018

The total number of points is 36 points. In order to get the grade 5.5 you will need at most 18 points. No notes and calculators are allowed. **Explain each step in your solution.** This exam has 6 tasks.

Task 1 (Gershgorin circles) (3 + 1 + 2 points)
Consider the matrix

$$A := \begin{pmatrix} -10 & 1 & 3 \\ 1 & 4 & 1 \\ 3 & 1 & 9 \end{pmatrix}.$$

a) Use the Gershgorin circles to approximately locate the eigenvalues of A. You do not have to use a similarity transform.

b) Illustrate your findings from a) in a picture.

c) Use the Gershgorin circle theorem to determine an upper bound for the condition number of the matrix A. Recall to that end that the condition number of A is given by $\kappa(A) = \|A\|_2\|A^{-1}\|_2$ and note that A is symmetric.

Task 2 (Eigenvalue solvers) (10 points)
Assume that you have to approximate the eigenvalue of largest magnitude of a large, sparse Hermitian matrix $A \in \mathbb{C}^{n\times n}$. Moreover, assume that λ_1 is a dominant eigenvalue, i.e.

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|.$$

Which algorithms that we discussed during the course can be recommended to compute such an approximation and why? Compare the key ideas of those (two) algorithms. Under which conditions can one be preferred over the other; consider the convergence behavior/rate and the computational costs in this context.

Task 3 (Galerkin method) (3 points)
Name three Galerkin methods; at least one should be an eigenvalue solver.

Task 4 (Fixed-point iteration) (4 + 1 + 2 points)
Consider the following function:

$$f(x) = \frac{1}{8}\cos(x) + \frac{1}{8}x^3.$$

(a) Use the Banach fixed-point theorem to show that f has a unique fixed-point in $[-1, 1]$.

(b) Give a numerical method to approximate the fixed-point of f.

(c) Choose $x_0 = 0$ as a starting point. We aim at achieving $\|\bar{x} - x_k\|_2 \leq 2^{-10}$, where $\bar{x}$ denotes the fixed-point. Use an a priori error estimate to derive an upper bound for the number of required iterations in the fixed-point iteration.

Task 5 (PageRank) (4 points)

Denote with $w_1, \dots, w_n$ all webpages in the considered network. Collecting the PageRanks of those webpages in a vector $\mathbf{R}$ we may formulate the following system of equations for their PageRank:

$$\mathbf{R} = d(P + A)\mathbf{R} + o\frac{1-d}{n}, \tag{1}$$

where $o = (1, \dots, 1)^T$,

$$P_{ij} = \begin{cases} 1/(C(w_j)) & \text{if } w_j \text{ points to } w_i, \\ 0 & \text{else} \end{cases} \quad \text{and} \quad A_{:,j} = \begin{cases} 1/n & \text{if the } j-\text{th column of P is 0}, \\ 0 & \text{else} \end{cases} \tag{2}$$

and $A_{:,j}$ denotes the j-th column of A and $C(w_j)$ denotes the number of links going out of page w_j. If we scale $\mathbf{R}$ such that $o^T\mathbf{R} = 1$ (1) is equivalent to the problem:

$$\mathbf{R} = (d(P + A) + oo^T\frac{1-d}{n})\mathbf{R}. \tag{3}$$

The Simplified PageRank algorithm just considers the following system of equations:

$$\mathbf{R} = dP\mathbf{R}. \tag{4}$$

Explain which problems occur with the Simplified PageRank algorithm if the network contains (at least) two disconnected subnetworks. Moreover, elaborate on how this problems are solved with the PageRank algorithm (1). Illustrate your discussion with a small graph and the corresponding P matrix.

Task 6 (Probabilistic a posteriori norm estimator) (6 points)

To estimate the spectral norm of the projection error $\|A - UU^TA\|_2$ for $A \in \mathbb{R}^{m\times n}$ and $U \in \mathbb{R}^{m\times k}$ orthonormal, we define the a posteriori norm estimator Δ for n_t mutually independent standard Gaussian test vectors ω_i, $i = 1, \dots, n_t$ as

$$\Delta := c_{\text{est}}(n_t, \delta_{\text{testfail}}) \max_{i\in 1,\dots,n_t} \|(A - UU^TA)\omega_i\|_2, \tag{5}$$

where $c_{\text{est}}(n_t, \delta_{\text{testfail}})$ is defined as $c_{\text{est}}(n_t, \delta_{\text{testfail}}) := 1/[\sqrt{2}\,\text{erf}^{-1}(\sqrt[n_t]{\delta_{\text{testfail}}})]$, and the error function erf is defined as $\text{erf} := \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}\,dt$. Note that the error function has the following interpretation: for $x \geq 0$ $\text{erf}(x)$ describes the probability that a normal random variable with mean zero and variance $1/2$ lies in the interval $[-x, x]$. Prove that Δ is an upper bound of $\|A - UU^TA\|_2$ with probability greater or equal than $(1 - \delta_{\text{testfail}})$. You can use without proof that the normal distribution is invariant under rotations, that means if we multiply a standard Gaussian vector ω with an orthonormal matrix U that $U\omega$ is again a standard Gaussian random vector.

overview points

1	**2**	**3**	**4**	**5**	**6**
6	10	3	7	4	6

Table 1: Total: 36 points