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Exam for the Course

Scientific Computing

Quartile 3 2017/2018

The total number of points is $40+4$ points. In order to pass the exam you will need at most 24 points. No notes and calculators are allowed. **Explain each step in your solution.** This exam has 8 tasks.

Task 1 (LU decomposition) (4 + 2 points)
Consider the linear system of equations $Ax = b$ with

$$A = \begin{pmatrix} 6 & 4 & 4 \\ 3 & 4 & 6 \\ 6 & 6 & 4 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{1}$$

a) Compute the LU decomposition of A defined in (1) without pivoting.

b) Use the just determined LU decomposition of the matrix A defined in (1) to solve the linear system of equations $Ax = b$.

Task 2 (QR decomposition via Householder reflections) (3 points)
Let $v \in \mathbb{R}^n$ and $I \in \mathbb{R}^{n\times n}$. Explain why the Householder matrix

$$H(v) := I - 2\frac{vv^T}{v^Tv}$$

is a reflection. Draw a picture to illustrate your explanation.
Hint: Apply the matrix $H(v)$ to a vector.

Task 3 (least squares problem) (5 points)
Compute the line of best fit $f(x) = a + bx$ for the following data points by making use of the normal equations.

x_i	-3	-2	-2	1
$f(x_i)$	-0.7	1.5	1.1	1.1

Task 4 (SVD) (6 points)
For $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ we have given a singular value decomposition $A = U\Sigma V^T$ with $V^TV = I_{n\times n}$, $U^TU = I_{m\times m}$ and $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{m\times n}$. For r, $1 \leq r < n$ we define an approximation $A_r := U\Sigma_r V^T$ with $\Sigma_r := \operatorname{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0) \in \mathbb{R}^{m\times n}$. Show that for $r = 1$ we have

$$\inf_{B\in\mathbb{R}^{m\times n}, \operatorname{rang}(B)=r} \|A - B\|_F = \|A - A_r\|_F, \tag{2}$$

where the Frobenius norm is defined as $\|A\|_F := (\sum_{i,j=1}^{m,n}(a_{ij}^2))^{1/2}$.
You may use without proof that ...

- ... a matrix $B \in \mathbb{R}^{m\times n}$ with rank 1 can be written as $B = \sigma_1' \cdot u \cdot v^T$ with $\sigma_1' > 0$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, $\|u\|_2 = \|v\|_2 = 1$
- ... there holds $\|A\|_F = \sqrt{\operatorname{trace}(A^TA)} = (\sum_i^n \sigma_i^2)^{1/2}$,
- ... the Frobenius norm is invariant under orthogonal transformations.

Task 5 (Comparison of iterative solvers for linear systems of equations) (8 points)
Compare the method of steepest descent with the conjugate gradient method. Include the following aspects in your discussion:

- Short description of the key idea of the respective method.
- Choice of the search direction. (Precise formula for the conjugate gradient method is not needed; description of the idea on how to choose the search direction is sufficient)
- To which class of iterative solver methods does the respective method belong to?
- Convergence rate of the respective method.

Task 6 (Weak derivative) (4 points)
Determine whether the following function has a weak derivative and if it does compute it. Explain your answer in full detail.

$$g(x) = \begin{cases} 2 - x \text{ if } 0 \le x \le 1, \\ \log_e(x) + 1 \text{ if } 1 < x \le 2. \end{cases}$$

Task 7 (Existence of weak solutions) (2 points)
Explain why it is often neccessary to consider the weak formulation of a partial differential equation instead of the strong formulation.

Task 8 (Céa Lemma) (2 + 4 points)
State and prove the Céa Lemma. If you do not remember the statement you can obtain the statement of the Céa Lemma in exchange for 2 points and then do the proof.

overview points

1	2	3	4	5	6	7	8
6	3	5	6	8	4	2	6

Table 1: Total: 40 + 4 points