Department of Applied Mathematics
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June 12, 2020, 13:45–16:45

Exam for the Course

Scientific Computing (code: 191551200)

2019/2020

The exam is held based on the assumption that you will do it yourself, without the help from other people. You are only allowed to use

- The lecture notes (including your own annotations to the lectures notes (that means that if you have written something in the pdf or printed version)).
- Your own notes you took during the course.
- All slides and videos available on the Canvas page of the course Scientific Computing 2019/2020 (course code 191551200) at the University of Twente.
- The book Quarteroni, Alfio, Sacco, Riccardo, Saleri, Fausto. Numerical Mathematics, Second Edition, Springer, Berlin Heidelberg New York, 2007.
- The exercise sets and your solutions.
- The practical assignments and your solutions.

Please read the following paragraphs carefully, and copy the text below in italics to your answer sheet. To find more information, please consult `https://canvas.utwente.nl/courses/4925/discussion_topics/58074`. By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.
Text to be copied: *I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.*

Sign this statement, and write below your signature your name, student number and study program.

Students have to submit their solutions, a picture of their student card, and, if applicable, the card allowing extra time as one single pdf file. Other formats than pdf will not be accepted. Please make sure that the scan is clear and good to read. What we cannot read, we cannot grade.

Students will have 20 minutes after the exam is finished to produce this single pdf file and upload it to Canvas; hence till 17:05 for regular students and 17:50 for students with permission for extra time. After this, the site will be closed and no answers will be accepted.

The grades obtained from the exam will not automatically be the officially registered, definite grades. Instead, the grades first enter an intermediate stage of “crowned grades”, before becoming “definite grades”.

In case we observe significant irregularities, this will be reported to the Examination Board and at least the grade might not be turned into a “definite grade”.

In case a picture of the student card, a picture of the card allowing extra time, or the signed integrity statement is missing, this will be reported to the Examination Board and the grade might not be turned into a “definite grade”.

Please indicate clearly all the questions you solve and avoid writing your solution on different, disjoint pages.

All answers must be motivated and clearly formulated. Explain each step in your solution. Your solutions should make very clear to us that you understand all of the steps and the logic behind the steps.

The total number of points is 54 points. In order to get the grade 5.5 you will need at most 27 points. This exam has 7 tasks.

Task 1 (Singular value decomposition) (7 + 5 + 2 points)
Consider the set of matrices

$$\mathcal{A}_{2\times 2} := \left\{A \in \mathbb{R}^{2\times 2} \,:\, A_{ij} \in \{0,1\}\right\} \setminus A_{00}, \quad \text{where} \quad A_{00} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

a) Consider now the singular values of all matrices in $\mathcal{A}_{2\times 2}$. What is the slowest and (close to) fastest decay of the singular values you expect for all matrices in $\mathcal{A}_{2\times 2}$ (here we define the decay as $\sigma_1 - \sigma_2$)? Provide one matrix A_s that realizes this slowest decay and one matrix A_f that realizes the (close to) fastest decay. Elaborate on your answers and explain your choice. Simply providing the matrices and the decay of the singular values without explaining how you got to your answer will result in zero points.

b) Compute $A_{*,1} = \sigma_{*,1} u_{*,1} v_{*,1}^T$ for both $* = s$ and $* = f$, where $u_{*,1}$ and $v_{*,1}$ are the first column of U_* and V_*, respectively, of $A_* = U_* \Sigma_* V_*^T$, $* = s, f$. Provide the error $\|A_* - A_{*,1}\|_2$ and reflect on the results.

c) Provide for both A_s and A_f a picture in which you draw $A_* v_{*,1}$ and $A_* v_{*,2}$, for $* = s, f$.

Task 2 (Comparison of solvers for linear systems of equations) (10 points)
We are given the task to solve (approximately) a linear system of equations $Ax = b$, where the matrix $A \in \mathbb{R}^{n\times n}$ is large, sparse, symmetric, and positive definite. Which algorithms that we discussed during the course can be recommended to compute such an approximation and why? Comment also briefly why other methods we discussed are not suited for the current task. Compare the (two) algorithms that are suited for the task in terms of ...

- their key ideas (precise formulas are not needed; state also to which class of iterative solvers the respective method belongs to)
- their convergence behavior/rate
- their computational costs.

Based on your comparison decide which method you would recommend using.

Task 3 (Weak formulation) (4 + 3 points)
Let $I = (a, b)$, $f \in L^2(I)$, $p \in C^0(\bar{I})$, $p(x) \geq p_0 > 0$, $\alpha \in C^1(\bar{I})$, $\beta \in C^0(\bar{I})$ and define the vector space $X := \{v \in H^1(I) \,:\, v(a) = 0\}$ equipped with the H^1-norm (X is the same as in Chapter 4.1 in the lecture notes). Moreover, we introduce the bilinear form $b : X \times X \to \mathbb{R}$ defined as

$$b(u, v) := \int_I p(x) u'(x) v'(x)\, dx + \int_I \alpha(x) u'(x) v(x)\, dx + \int_I \beta(x) u(x) v(x)\, dx. \tag{1}$$

Let $y \in X$ satisfy

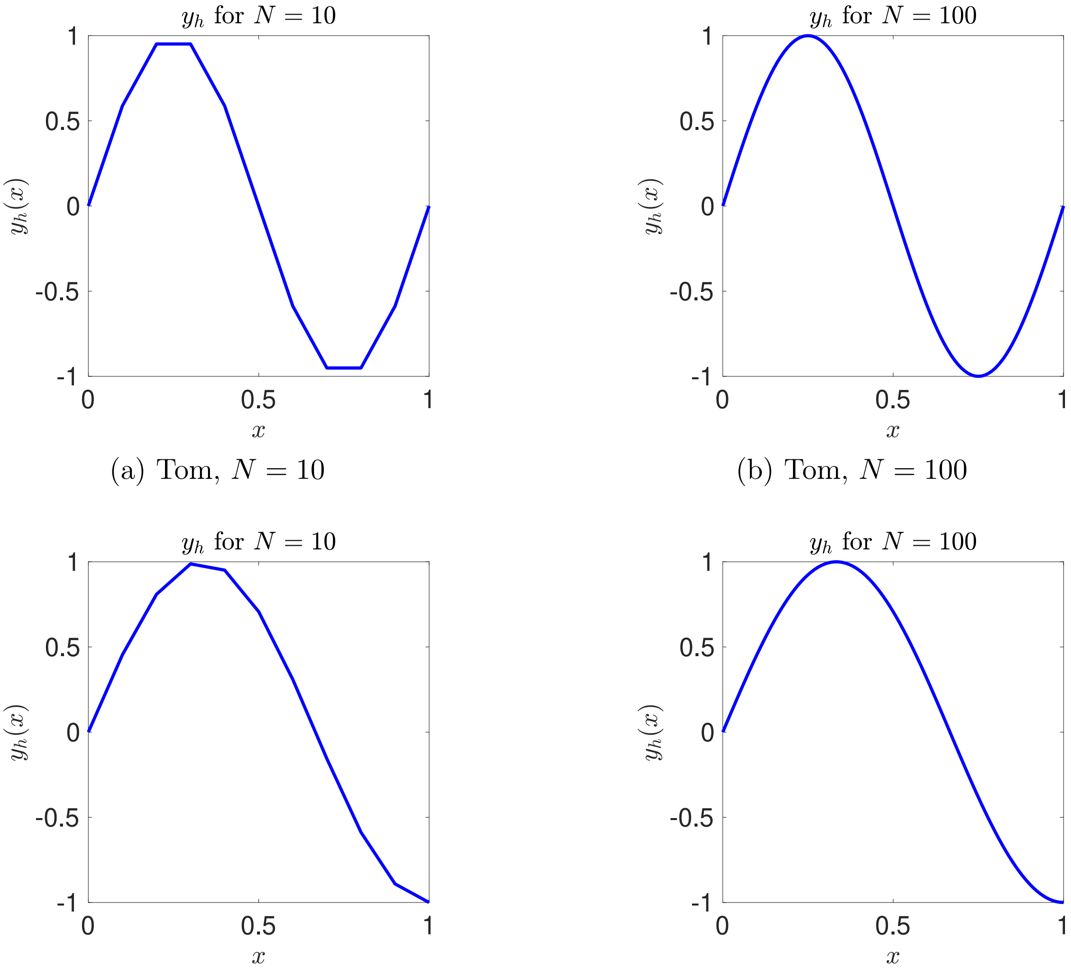
$$b(y, v) = \int_I f(x) v(x)\, dx \quad \forall v \in X. \tag{2}$$

a) Derive conditions on $\alpha(x)$ and $\beta(x)$ different from the trivial cases $\alpha \equiv 0$ or $\beta \equiv 0$ such that the bilinear form is coercive with respect to the norm $\|v\|_X := \left(\int_I (v'(x))^2\, dx\right)^{1/2}$. The conditions must not depend on $p(x)$ or p_0. Hint: Perform partial integration on the term $\int_I \alpha(x) u'(x) v(x)\, dx$.

b) Prove that under the conditions on $\alpha(x)$ and $\beta(x)$ you derived in a) there holds

$$\|y\|_X \leq C\, \|f\|_{L^2(a,b)}$$

for a constant $0 < C < \infty$.



(a) Tom, $N = 10$ (b) Tom, $N = 100$

(c) Jerry, $N = 10$ (d) Jerry, $N = 100$

Figure 1: Finite element approximations.

Task 4 (Finite Element approximation) (4 points)

Let $I = (0, 1)$ and define the vector space $X := \{v \in H^1(I) : v(0) = 0\}$ equipped with the H^1-norm. Moreover, let $y \in X$ be the weak solution of the problem, satisfying

$$\int_I y'(x)v'(x)\,dx = \int_I 2.25\pi^2 \sin(1.5\pi x)v(x)\,dx \quad \forall v \in X. \tag{3}$$

Additionally, let $I_h := \{x_0, \ldots, x_N\} \subset I$ be a mesh with $x_0 = 0,\ x_N = 1,\ x_{j+1} = x_j + h,\ h > 0$ and define $I_j := (x_{j-1}, x_j)$, $j = 1, \ldots, N$. Finally, let $X_h \subset X$ be the linear Finite Element space

$$X_h := \{\phi_h \in C^0([0, 1]) \,|\, \phi_h(0) = 0, \phi_h|_{I_j} \in \mathbb{P}_1\ \forall j = 1, \ldots, N\}$$

and $y_h \in X_h$ the Finite Element approximation of the weak solution $y \in X$ of (3). Two students Tom and Jerry implemented an algorithm to obtain y_h for $N = 10$ and $N = 100$. Reflect on the plots they obtained, which are depicted in Fig. 1.

Task 5 (Gershgorin circles) (3 + 1 + 2 points)

Consider the matrix

$$A := \begin{pmatrix} 10 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

a) Use the Gershgorin circles to approximately locate the eigenvalues of A. You do not have to use a similarity transform.

b) Illustrate your findings from a) in a picture.

c) Use the Gershgorin circle theorem to determine an upper bound for the condition number of the matrix A.

Task 6 (Concentration inequalities) (7 points)
Let X_i be independent Bernoulli random variables with parameters p_i. Recall that a Bernoulli random variable X_i with parameter p_i satisfies $Pr(X_i = 1) = p_i$, $0 \leq p_i \leq 1$ and $Pr(X_i = 0) = 1 - p_i$. Consider their sum $S_N = \sum_{i=1}^{N} X_i$ and denote its mean by $\mu = \mathbb{E}S_N$, noting that there holds $\mu = \sum_{i=1}^{N} p_i$. Prove that for any $t > \mu$ there holds

$$Pr(S_N \geq t) \leq e^{-\mu} \left(\frac{e\mu}{t}\right)^t .$$

Explain each step briefly. Hint: Proceed as in the proof of Theorem 7.13 (Hoeffding's inequality) in the lecture notes. You may use without proof that for all $x \in \mathbb{R}$ we have $1 + x \leq e^x$.

Task 7 (Mixed-integer Programming Model) (6 points)
Michael plans his next *Darts* tour through Europe. He considers n cities $V := \{1, 2, \ldots, n\}$ for a potential visit. His trip will start at Vlijmen, denoted by 0, continues with a number of city visits and ends at Vlijmen. Each of the cities V is visited at most once. Let $V_0 := V \cup \{0\}$ denote all relevant locations.
Traveling from location $i \in V_0$ to another location $j \in V_0$ takes $t_{i,j} > 0$ hours and a visit of city $i \in V$ takes $d_i > 0$ hours. You can assume that the triangle inequality for the travel times t is satisfied. Unfortunately, he won't be able to visit all cities due to lack of time: Michael has a total time budget of $T > 0$ hours that he can spend for the whole trip, which includes the time required for returning home.
He receives $p_i > 0$ Euros if we visits city $i \in V$, but has to pay $c_{i,j} > 0$ Euros for traveling from $i \in V_0$ to $j \in V_0$. His goal is to maximize the profit, i.e., the sum of all payments he receives minus the money he has to spend for traveling.
Finally, in order to satisfy fans in all countries, he wants to visit at least one city in every country. For this, we partition the set of cities as $V = C_1 \cup C_2 \cup \ldots \cup C_k$, where k denotes the number of countries and C_ℓ contains the cities from country $\ell \in \{1, 2, \ldots, k\}$. Note that there is no limit on crossing borders, i.e., he does not have to visit the cities in a country consecutively.
Create a mixed-integer program to help Michael solve this optimization problem. Briefly explain the meaning of your variables and constraints (one phrase or sentence each).

overview points

1	**2**	**3**	**4**	**5**	**6**	**7**
14	10	7	4	6	7	6

Table 1: Total: 54 points