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June 30, 2023

Exam for the Course

Scientific Computing

2022/2023 – Exam – 13:45-16:56

The use of electronic devices or notes is not allowed. All of your answers need to be justified. Good luck!

Exercise 1 (LU decomposition) (4 + 2 Points)

We have given the following matrix A:

$$A := \begin{pmatrix} a & 0 & 2 \\ 4 & -4 & 5 \\ 3 & 4 & c \end{pmatrix} \qquad \text{with } a, c \in \mathbb{R}. \tag{1}$$

a) Compute the LU decomposition of A defined in (1) without pivoting, where L has to be a unit lower triangular matrix. Check whether your LU decomposition satisfies indeed $A = LU$. For this subtask you can assume that a, c are chosen such that a unique LU decomposition exists.

b) Give conditions for a, c that are sufficient for the existence of a unique LU decomposition with a unit lower triangular matrix L and an upper triangular matrix U such that $A = LU$ for A defined in (1). Give also a brief explanation.

Exercise 2 (Moore-Penrose pseudo-inverse/SVD) (2+4 Points)

a) Let $A \in \mathbb{C}^{m\times n}$, $m \geq n$ with $\operatorname{rank}(A) = n$ and let $A = U\Sigma V^*$ be the singular value decomposition of A with $U \in \mathbb{C}^{m\times m}, \Sigma \in \mathbb{C}^{m\times n}, V \in \mathbb{C}^{n\times n}$. Show that the Moore-Penrose pseudo-inverse $A^\dagger := V\Sigma^\dagger U^*$ is given by

$$A^\dagger = (A^*A)^{-1}A^*.$$

b) Compute the thin singular value decomposition for

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 3 \\ 1 & -1 \end{pmatrix}.$$

Exercise 3 (Weak derivative/Céa Lemma) (2+3 Points)

a) Determine whether the following function has a weak derivative and if it does compute it. Explain your answer in full detail.

$$g(x) = \begin{cases} 2 - x & \text{if } 0 \le x \le 1, \\ \ln(x) + 1 & \text{if } 1 < x \le 2. \end{cases}$$

b) Let X be a Hilbert space and let the bilinear form $b : X \times X \to \mathbb{R}$ be symmetric, continuous (with continuity constant C_B) and X-elliptic (with constant C_E). Additionally, assume that for any subspace X_h of X we have $u \in X$ and $u_h \in X_h$ such that

$$\begin{aligned} b(u, v) &= g(v) \qquad \forall v \in X, \\ b(u_h, v_h) &= g(v_h) \qquad \forall v_h \in X_h. \end{aligned}$$

Give the definition of X-elliptic and continuity for bilinear form $b : X \times X \to \mathbb{R}$. Using these properties, then prove the following estimate

$$\|u - u_h\|_X \le \frac{C_B}{C_E} \inf_{v_h \in X_h} \|u - v_h\|_X .$$

Exercise 4 (Gershgorin circles) (3 + 1 + 2 Points)

Consider the matrix

$$A := \begin{pmatrix} 6 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 10 \end{pmatrix}.$$

a) Use the Gershgorin circles to approximately locate the eigenvalues of A.

b) Illustrate your findings from a) in a picture.

c) Use the Gershgorin circles to determine an upper bound for the condition number of the matrix A with respect to the $\|\cdot\|_2$-norm. Recall to that end that $\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2$ and note that A is real and symmetric.

Exercise 5 (PageRank) (4 Points)

Denote with $w_1, \dots, w_n$ all webpages in the considered network and with $PR(w_i)$ the PageRank of page w_i, $i = 1, \dots, n$. We denote by the vector $\mathbf{x} := (PR(w_1), \dots, PR(w_n))^T$ the vector of all PageRanks. Then, we may formulate the following system of equations to compute $\mathbf{x}$

$$\mathbf{x} = \left[d(A + R) + e^T e \frac{1-d}{n} \right] \mathbf{x}, \qquad \text{where } e = (1, \dots, 1)^T, \tag{2}$$

$$a_{ij} = \begin{cases} 1/(C(w_j)) & \text{if } w_j \text{ points to } w_i, \\ 0 & \text{else} \end{cases} \quad \text{and } R_{:,j} = \begin{cases} 1/n & \text{if the } j-\text{th column of A is 0.} \\ 0 & \text{else} \end{cases} \tag{3}$$

Here, we denote by $R_{:,j}$ the j-th column of R and by $C(w_j)$ we denote the number of links going out of page w_j. The Simplified PageRank algorithm just considers the following system of equations:

$$\mathbf{x} = A\mathbf{x}. \tag{4}$$

Explain which problems occur with the Simplified PageRank algorithm if the network contains dangling nodes. Moreover, elaborate on how these problems are solved with the PageRank algorithm (2). Illustrate your discussion with a small graph and the corresponding A matrix. (Hint: Consider the properties of a stochastic matrix.)

Exercise 6 (Randomized SVD) (3 Points)
Provide the algorithm to perform a randomized SVD of a given matrix $A \in \mathbb{R}^{m\times n}, m \geq n$. Assume that you already have computed a matrix $U \in R^{m\times(k+p)}$ (where $k+p \leq n$) with $k+p$ orthonormal columns such that

$$\|A - UU^T A\| \leq \text{tol}.$$

Given A and U write down the remaining steps and also the randomized SVD factorization of A.

Exercise 7 (Mixed-integer Programming Model) (6 Points)
Eni wants to schedule the flour production of her mill for a day. She can produce different flour types $K := \{1, 2, \dots, n\}$, and for each $k \in K$ she has a demand of $d_k^1 \in \mathbb{Z}_{\geq 0}$ at noon (after 720 potential production minutes) and an additional demand of $d_k^2 \in \mathbb{Z}_{\geq 0}$ at midnight (after another 720 potential production minutes). Thus, the total demand of flour $k \in K$ is $d_k^1 + d_k^2$, for which she may decide to produce more than d_k^1 in the morning and less than d_k^2 in the afternoon. The production times are given as $p_k \in \{5, 10, 15, 20, \dots\}$. However, whenever the flour type is changed the mill needs 10 minutes reconfiguration time in which nothing can be produced. At the beginning, no reconfiguration is necessary.

Create a mixed-integer program to help Eni solve this feasibility problem! Briefly explain the meaning of your variables and constraints (one phrase or sentence each).

Exercise	1.	2.	3.	4.	5.	6.	7.	total	Grade:
Points	4+2	3+4	2+3	3+1+2	4	2	6	36	(Points + 4)/4