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Exam for the Course

Scientific Computing

2022/2023 – Resit – 08:45-11:45

The use of electronic devices or notes is not allowed. All of your answers need to be justified. Good luck!

Exercise 1 (LU decomposition) (3+1+2 points)

We have given the following matrix A:

$$A := \begin{pmatrix} 2 & -1 & -3 \\ 4 & 0 & -3 \\ 6 & 1 & -1 \end{pmatrix}.$$

a) Compute the LU decomposition (without pivoting) where the diagonal entries of L equal 1 of the matrix A. Write down all intermediate steps. Do not forget to check after you have determined L and U whether L, U, and A actually satisfy $A = LU$.

b) Compute the determinant of A by employing its LU decomposition.

c) Use the LU decomposition to compute the solution of $Ax = b$ for $b = (-2, -9, -22)^T$.

Exercise 2 (Moore-Penrose pseudo-inverse/SVD) (4 Points)

Let $A \in \mathbb{C}^{m\times n}$ with rank $r \leq n$ and SVD given by $A = U\Sigma V^*$ with $U \in \mathbb{C}^{m\times m}, \Sigma \in \mathbb{C}^{m\times n}, V \in \mathbb{C}^{n\times n}$. Show that the unique solution to the problem: Find $\tilde{x} \in \mathbb{C}^n$ with minimal Euclidian norm such that

$$||A\tilde{x} - b||_2^2 \leq \min_{x\in\mathbb{C}^n} ||Ax - b||_2^2,$$

is given by $\tilde{x} = A^\dagger b$, where $A^\dagger := V\Sigma^\dagger U^*$ is the Moore-Penrose pseudo-inverse.

Exercise 3 (Weak derivative/Lax Milgram) (2+3+1 Points)

a) Determine whether the following function

$$g(x) = |\sin(x)| \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

has a weak derivative and if it does compute it. Explain your answer in full detail.

b) Give the definition of symmetry, X-elliptic and continuity for bilinear form $b : X \times X \to \mathbb{R}$. Moreover, prove uniqueness of solution $u \in X$ in the Lax-Milgram Theorem (provided in the following).
(**Theorem: Lax-Milgram**) Let X be a Hilbert space and the bilinear form $b : X \times X \to \mathbb{R}$ be symmetric, continuous and X-elliptic. Moreover, let $g : X \to \mathbb{R}$ be a continuous linear form.

Then, there exist a unique solution $u \in X$, such that

$$b(u, v) = g(v) \text{ holds for all } v \in X.$$

Moreover, $u \in X$ is the minimizer of

$$J(v) = \frac{1}{2} b(v, v) - g(v)$$

under all $v \in X$.

c) Additionally, assume that for any subspace X_h of X we have $u \in X$ and $u_h \in X_h$ such that

$$\begin{aligned} b(u, v) &= g(v) & \forall v \in X, \\ b(u_h, v_h) &= g(v_h) & \forall v_h \in X_h. \end{aligned}$$

Show the Galerkin orthogonality relation.

Exercise 4 (Gershgorin circles) (4 + 2 points)
Consider the matrix

$$A := \begin{pmatrix} 3 & -1/2 & 0 \\ 2 & 1 & 0 \\ -1 & 1/2 & 8 \end{pmatrix}.$$

a) Use the Gershgorin circles to approximately locate the eigenvalues of A. Additionally, determine the eigenvalue of A that has the largest absolute value. Explain each step.

b) Illustrate your findings from a) in a picture.

Exercise 5 (Power Method) (3+1+1 points)

a) What does the Power method approximate? In addition, write down the algorithm of the power method.

b) To derive convergence, we needed that the matrix A is diagonalizable and that λ_1 must be a dominant eigenvalue of A. Explain what is meant by these two concepts.

c) How do the Krylov subspace methods improve over the power method?

Exercise 6 (Element-wise sampling) (1,5 + 1,5 points)

Let $A \in \mathbb{R}^{m \times n}$. We use element-wise sampling to construct a random sketch $\tilde{A} \in \mathbb{R}^{m \times n}$ of A.

a) Explain why performing uniform sampling, i.e. setting $p_{ij} = 1/(mn)$ for all i, j is in general not a good idea. Moreover, give an improved sampling distribution p_{ij}, which still samples element-wise.

b) For a quadratic matrix $A \in \mathbb{R}^{n \times n}$ an error estimate for the random sketch $\tilde{A}$ resulting by element-wise sampling is given by

$$\|A - \widetilde{A}\|_2 \leq \mathcal{O}\left(\left[\frac{n \log(n)}{s} \right]^{1/2} \right) \|A\|_F,$$

with probability at least $1 - 1/n$. The number $s \in \mathbb{N}$ denotes the number of drawn random samples. Explain why element-wise sampling does in general require a large number of samples s to obtain satisfactory results.

Exercise 7 (Mixed-integer Linear Programming Model) (6 Points)

Rob wants to plan the running time of a power plant. He can decide to turn it off after any day $t \in \{1, 2, \ldots, T\}$ from now on, where $T \in \mathbb{Z}$ is a given integer. The power plant is running right now, and once it is turned off it will remain so, and not produce any energy. For each day t it runs he can decide how much energy $e_t \in \mathbb{R}$ it shall produce, whereas $e_0 > 0$ is a given parameter that states the produced energy on day 0. However, there is a minimum amount $\ell > 0$ of energy that it must produce on any day (for technical reasons). Moreover, for each day t there also is a maximum amount $u_t > \ell$ of energy it is allowed to produce (to not overload the net). The production increase per day is limited by $\Delta > 0$, i.e., the productions e_t and e_{t+1} must satisfy $e_{t+1} \leq e_t + \Delta$. This also affects the increase from day 0 to day 1. The previous restrictions only apply if the power plant is running since otherwise e_t will be equal to 0. Moreover, we have a prediction of the profit $p_t \in \mathbb{R}$ per unit of energy produced in t as well as costs $c_t \in \mathbb{R}$ for running it. There are neither costs not profit once the power plant is turned off. Create a mixed-integer *linear* program to help Rob solve the problem of maximizing total estimated profit (minus the costs)! Briefly explain the meaning of your variables and constraints (one phrase or sentence each).

Exercise	1.	2.	3.	4.	5.	6.	7.	total	Grade:
Points	3+1+2	4	2+3+1	4+2	3+1+1	1,5+1,5	6	36	(Points + 4)/4